Full Counting Statistics for the Kondo Dot

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The generating function for the cumulants of charge current distribution is calculated for two generalized Majorana resonant level models: the Kondo dot at the Toulouse point and the resonant level embedded in a Luttinger liquid with the interaction parameter $g = 1/2$. We find that the low-temperature nonequilibrium transport in the Kondo case occurs via tunneling of physical electrons as well as by coherent transmission of electron pairs. We calculate the third cumulant ("skewness") explicitly and analyze it for different couplings, temperatures, and magnetic fields. For the $g = 1/2$ setup the statistics simplifies and is given by a modified version of the Levitov-Lesovik formula.

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Since Schottky's realization that the shot noise in a conductor contains invaluable information about the physical properties of the charge carriers, the question about the noise spectra of different circuits became as important as the knowledge of their current-voltage characteristics [1]. The noise constitutes the second moment of the current distribution function (which is the probability of measuring a given value of the current) and is supposed to contain information about the charge of current carrying excitations at weak transmission (reflection). That is indeed the case for $S-S$ and $S-N$ junctions [2–4] but has not been proven for a generic interacting model. As has been pointed out in [5], the third cumulant also contains valuable information about the charge of the current carriers. Therefore, it is natural to investigate the full current distribution function. This was an academic question for a very long time as even the measurement of the second cumulant remained on the frontier of experimental physics. Only after the work by Reulet and co-workers, the measurement of the third cumulant became possible [6]. Inspired by this remarkable achievement, the full current distribution function (more often referred to as "full counting statistics" or FCS) has been theoretically analyzed in recent years for a wide range of systems.

In their seminal work [7], Levitov and Lesovik derived the exact formula for the FCS for the electron tunneling setup (single channel):

$$\ln\chi_0(\lambda;V;\{T(\omega)\}) = \mathcal{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln\{1+T(\omega)[n_L(1-n_R)\times(e^{i\lambda}-1)+n_R(1-n_L)(e^{-i\lambda}-1)]\}, \quad (1)$$

where $\mathcal{T}$ is the waiting time, λ is the measuring field (we will explain this notation in more detail shortly), $n_{R(L)}(\omega) = n_F(\omega \pm V/2)$ are the electron filling factors in the (right and left) leads, V is the bias voltage, and $T(\omega)$ is the single electron transmission coefficient. The knowledge of $T(\omega)$ thus fully defines the FCS for noninteracting systems.

While the Levitov-Lesovik approach can be relatively easily generalized to various multiterminal (multichannel) setups, it is notoriously difficult to include electron-electron interactions. Up to now most works in this direction relied upon various perturbative expansions, in the tunneling amplitude [5,8] or in the interaction strength [9] (for a recent review, see [10]), as well as calculations at zero temperature [11]. Certainly no paradigm for an interacting FCS emerged as of yet. Notable exceptions are the contribution [12] as well as recent works by Andreev and Mishchenko and Kindermann and Trauzettel [13,14], where the exact FCS was calculated for the (single-channel) Coulomb blockade (CB) setup of Matveev and Furusaki [15,16]. We shall establish the precise connection of these works to our results.

The purpose of this Letter is to contribute to our understanding of interacting FCSs by means of obtaining the exact FCS for two particular experimentally relevant setups: the Kondo dot and the resonant tunneling (RT) between two $g = 1/2$ Luttinger liquids (LL).

We start with a brief description of the method. Our goal is the calculation of the generating function $\chi(\lambda) = \sum e^{iq\lambda}P_q$ for the probabilities P_q of q electrons being transmitted through the system over time $\mathcal{T}$. We first define operators T_i transferring one electron through the system in the direction of the current ($i = R$) and in the reversed direction ($i = L$). The electron counting operator on the Keldysh contour C can then be written down in its canonical form as [7]

$$T_\lambda = e^{i\lambda(t)/2}T_R + e^{-i\lambda(t)/2}T_L,$$

where the measuring field $\lambda(t)$ is explicitly time dependent, $\lambda(t) = \lambda\theta(t)\theta(\mathcal{T}-t)$ on the forward path, and $\lambda(t) = -\lambda\theta(t)\theta(\mathcal{T}-t)$ on the backward path. According to [5,7], the generating function is then given by the expectation value

$$\chi(\lambda) = \langle T_C e^{-i\int_C T_\lambda(t)dt}\rangle. \quad (2)$$

In order to calculate $\chi(\lambda)$ we define a more general functional $\chi[\lambda_-(t), \lambda_+(t)]$ formally given by the same Eq. (2) but where $\lambda(t)$ is now understood to be an arbitrary function on the Keldysh contour, $\pm$ referring to two different functions on the time and antitime ordered halves of the contour. Next we assume that both $\lambda_\pm(t)$ change slowly in time. Then, neglecting switching effects, one obtains at large $\mathcal{T}$

$$\chi[\lambda_-(t), \lambda_+(t)] = \exp\left\{-i\int_0^{\mathcal{T}} \mathcal{U}[\lambda_-(t), \lambda_+(t)]dt\right\}, \quad (3)$$

where $\mathcal{U}(\lambda_-, \lambda_+)$ is the *adiabatic potential*. Once the adiabatic potential is computed, the statistics is recovered from $\ln\chi(\lambda) = -i\mathcal{T}\mathcal{U}(\lambda, -\lambda)$. As $\lambda_\pm$ are external parameters, performing the derivative of both (3) and (2) with respect to, say, λ_-, with help of the Feynman-Hellmann theorem [17], we immediately obtain

$$\frac{\partial}{\partial\lambda_-}\mathcal{U}(\lambda_-, \lambda_+) = \left\langle\frac{\partial T_\lambda(t)}{\partial\lambda_-}\right\rangle_\lambda,$$

where we use notation

$$\langle A(t)\rangle_\lambda = \frac{1}{\chi(\lambda_-, \lambda_+)}\langle T_C\{A(t)e^{-i\int_C T_\lambda(t)dt}\}\rangle.$$

This is somewhat more complicated than the usual *Hamiltonian formalism* for a quasistationary situation; we will give further technical details in the long version [18]; in particular, we have verified that Eq. (1) comes out correctly in the noninteracting case.

In order to study the FCS for the Kondo dot we use the bosonization and refermionization approach, originally applied to this problem by Emery and Kivelson [19] (see also [20]) and refined by Schiller and Hershfield (SH) (see [21]). The starting point is the two-channel Kondo Hamiltonian (we set $\hbar = v_F = e = k_B = 1$ throughout),

$$H = H_0 + H_J + H_M + H_V,$$

where, with $\psi_{\alpha,\sigma}$ being the electron field operators in the R, L channels,

$$\begin{aligned} H_0 &= i\sum_{\alpha=R,L}\sum_{\sigma=\uparrow,\downarrow}\int dx\psi^\dagger_{\alpha\sigma}(x)\partial_x\psi_{\alpha\sigma}(x), \\ H_J &= \sum_{\alpha,\beta=R,L}\sum_{\nu=x,y,z} J^{\alpha\beta}_\nu s^\nu_{\alpha\beta}\tau^\nu, \\ H_V &= (V/2)\sum_\sigma\int dx(\psi^\dagger_{L\sigma}\psi_{L\sigma} - \psi^\dagger_{R\sigma}\psi_{R\sigma}), \\ H_M &= -\mu_B g_i H\tau^z = -\Delta\tau^z. \end{aligned} \quad (4)$$

Here $\tau^{\nu=x,y,z}$ are the Pauli matrices for the impurity spin and ($\alpha, \beta = R, L$; $\sigma =\uparrow, \downarrow$; $\sigma^\nu_{\sigma\sigma'}$ are the components of the νth Pauli matrix)

$$s^\nu_{\alpha\beta} = \sum_{\sigma,\sigma'}\psi^\dagger_{\alpha\sigma}(0)\sigma^\nu_{\sigma\sigma'}\psi_{\beta\sigma'}(0)$$

are the electron spin densities in (or across) the leads, biased by a finite V. The last term in Eq. (4) stands for the magnetic field, $\Delta = \mu_B g_i H$. Following SH, we assume $J^{\alpha\beta}_x = J^{\alpha\beta}_y = J^{\alpha\beta}_\perp$, $J^{LL}_z = J^{RR}_z = J_z$, and $J^{LR}_z = J^{RL}_z = 0$. The only transport process then allowed is the spin-flip tunneling, so that we obtain for the T_λ operator

$$\begin{aligned} T_\lambda = \frac{J^{RL}_\perp}{2}(&\tau^+ e^{i\lambda(t)/2}\psi^\dagger_{R\downarrow}\psi_{L\uparrow} + \tau^- e^{i\lambda(t)/2}\psi^\dagger_{R\uparrow}\psi_{L\downarrow} \\ &+ \tau^+ e^{-i\lambda(t)/2}\psi^+_{L\downarrow}\psi_{R\uparrow} + \tau^- e^{-i\lambda(t)/2}\psi^\dagger_{L\uparrow}\psi_{R\downarrow}). \end{aligned}$$

After bosonization, Emery-Kivelson rotation, and refermionization (see details in [21]) and going over to the Toulouse point $J_z = 2\pi$, which is the only approximation we make, one obtains

$$H' = H'_0 - i(J_- b\xi_f + J_+ a\eta_f) - i\Delta ab + T_\lambda, \quad (5)$$

where the counting term is given by

$$T_\lambda = -iJ_\perp b[\xi\cos(\lambda/2) - \eta\sin(\lambda/2)], \quad (6)$$

with $J_\pm = (J^{LL}_\perp \pm J^{RR}_\perp)/\sqrt{2\pi a_0}$, $J_\perp = J^{RL}_\perp/\sqrt{2\pi a_0}$ (a_0 is the lattice constant of the underlying lattice model), and a and b being local Majorana operators originating from the impurity spin. The fields η_f and ξ_f in the spin-flavor sector are equilibrium Majorana fields, whereas η and ξ in the charge-flavor sector are biased by V,

$$\begin{aligned} H'_0 = i\int dx[&\eta_f(x)\partial_x\eta_f(x) + \xi_f(x)\partial_x\xi_f(x) + \eta(x)\partial_x\eta(x) \\ &+ \xi(x)\partial_x\xi(x) + V\xi(x)\eta(x)]. \end{aligned} \quad (7)$$

Using Eqs. (5)–(7) one can straightforwardly evaluate the adiabatic potential $\mathcal{U}(\lambda_-, \lambda_+)$ as the problem has become quadratic in the Majorana fields.

Skipping details of the calculation, we report the resulting exact formula for the FCS of the Kondo dot, which is the *main result* of this Letter:

$$\begin{aligned} \ln\chi(\lambda) = \mathcal{T}\int_0^\infty\frac{d\omega}{2\pi}\ln\{&1 + T_1(\omega)[n_L(1-n_R)(e^{2i\lambda}-1) \\ &+ n_R(1-n_L)(e^{-2i\lambda}-1)] + T_2(\omega)[(n_F(1-n_R) \\ &+ n_L(1-n_F))(e^{i\lambda}-1) + (n_F(1-n_L) \\ &+ n_R(1-n_F))(e^{-i\lambda}-1)]\}, \end{aligned} \quad (8)$$

where now the filling factors are $n_{R,L} = n_F(\omega \pm V)$ [$n_F(\omega)$ being the conventional Fermi function], not to be confused with notation in Eq. (1). The "transmission coefficients" are

$$T_1 = \frac{\Gamma_\perp^2(\omega^2 + \Gamma_+^2)}{[\omega^2 - \Delta^2 - \Gamma_+(\Gamma_\perp + \Gamma_-)]^2 + \omega^2(\Gamma_+ + \Gamma_- + \Gamma_\perp)^2},$$

$$T_2 = \frac{2\Gamma_\perp\Gamma_-(\omega^2 + \Gamma_+^2) + 2\Delta^2\Gamma_\perp\Gamma_+}{[\omega^2 - \Delta^2 - \Gamma_+(\Gamma_\perp + \Gamma_-)]^2 + \omega^2(\Gamma_+ + \Gamma_- + \Gamma_\perp)^2},$$

where $\Gamma_i = J_i^2/2$ ($i = \pm, \perp$).

For small voltages and zero temperature, the generating function turns out to be quite simple,

$$\chi(\lambda) = [1 + T_e(e^{i\lambda} - 1)]^N, \tag{9}$$

where $N = \mathcal{T}V/\pi$ is the number of the incoming particles during the time interval $\mathcal{T}$ and $T_e = \sqrt{T_1(0)}$ is the effective transmission coefficient. We analyzed explicitly the behavior of the system around the Toulouse point. The trivialization (9) turns out to be robust against departure from this special point in parameter space [18]. While (9) is the standard Levitov-Lesovik result for spinful systems, the physical content of Eq. (8) is more interesting since it cannot be reduced to binomial statistics as in Eq. (9) at finite T. One can see that the charge current is carried by *two* different quasiparticles with charges $q_1 = 2e$ and $q_2 = e$. We identify the corresponding transmission coefficients as T_1 and T_2, respectively. It was realized by SH that at least in the case of finite magnetic field $\Delta \neq 0$ it can become energetically favorable to tunnel electron pairs through the impurity rather than single electrons [21]. The presence of the term containing 2λ in (8) can be interpreted as a signature of this effect. The full transport coefficient T_0 as calculated by SH turns out to be a *composite* one and it is recovered from $T_{1,2}$ through a very simple relation: $T_0 = T_1 + T_2/2$. From the point of view of the Kondo physics, the case when $T_2 = 0$ and the statistics reduces to a modified Levitov-Lesovik formula, $\chi(\lambda) = \chi_0^{1/2}(2\lambda, 2V, \{T_1(\omega)\})$ (binomial statistics at $T = 0$), is the symmetric model in zero field (the other, unphysical, case of $T_2 = 0$ is when $J_\pm = 0$). We have evaluated the first and the second cumulant of the Kondo FCS Eq. (8) which are the same as calculated by SH at all V and T [18]. We shall not reproduce these two cumulants here and concentrate instead on new results.

The full analytic expression for the third cumulant exists but is too lengthy to be given here. We shall rather investigate various limits and use numerics for the general case. So, at $T = 0$ we obtain:

$$\langle\delta q^3\rangle = \mathcal{T}\int_0^V \frac{d\omega}{2\pi}[T_2 + 8T_1 - 3(T_2 + 2T_1)(T_2 + 4T_1) + 2(T_2 + 2T_1)^3]. \tag{10}$$

In the zero magnetic field it yields the following limiting forms:

$$\langle\delta q^3\rangle_{V\to 0} \approx \mathcal{T}G_0\frac{2\Gamma_\perp\Gamma_-(\Gamma_- - \Gamma_\perp)}{(\Gamma_\perp + \Gamma_-)^3}V,$$
$$\langle\delta q^3\rangle_{V\to\infty} \approx \mathcal{T}\pi G_0\Gamma_\perp, \tag{11}$$

where $G_0 = 1/(2\pi)$ is the conductance quantum. At low voltages the cumulant is negative for $\Gamma_- < \Gamma_\perp$. Generally, under these conditions the nth cumulant appears to possess $n - 2$ zeroes as a function of V, according to numerics. The saturation value in the limit $V \to \infty$ is independent of the coupling in the spin-flavor channel because the fluctuations in the biased conducting charge-flavor channel are much more pronounced than those in the spin-flavor channel, which experiences only relatively weak equilibrium fluctuations.

In the opposite case of near equilibrium, all odd cumulants $\langle\delta q^{2n+1}\rangle$ are identically zero, which can readily be seen from Eq. (8) by substituting $n_{R,L} = n_F$ into it. In the limit of low temperatures $T \to 0$ we recover the conventional Johnson-Nyquist noise power [21]. Moreover, it can be shown that the leading behavior in temperature of *every* even order cumulant in this situation is linear; e.g., for $\langle\delta q^4\rangle$ we obtain

$$\langle\delta q^4\rangle \approx \mathcal{T}4G_0T\frac{\Gamma_\perp\Gamma_+(\Gamma_-\Gamma_+ + \Delta^2)}{[\Delta^2 + \Gamma_+(\Gamma_\perp + \Gamma_-)]^2}. \tag{12}$$

For the general situation of arbitrary parameters, the cumulants can be calculated numerically. The asymptotic value of the third cumulant at high voltages, similarly to the findings of [14], does not depend on temperature and is given by the result (11); see Fig. 1. In the opposite limit of small V, $\langle\delta q^3\rangle$ can be negative. Sufficiently large coupling Γ_- or magnetic field suppress this effect, though [18].

According to the result of Ref. [5], as long as the distribution is binomial, $\langle\delta q^3\rangle/\langle\delta q\rangle = (e^*)^2$, where e^* is the effective charge of the current carriers. This quantity is to be preferred to the Schottky formula because of its weak temperature dependence. Indeed we find numerically that the ratio $\langle\delta q^3\rangle/\langle\delta q\rangle$ in the present problem is weakly temperature dependent (it is flat and levels off to one) in

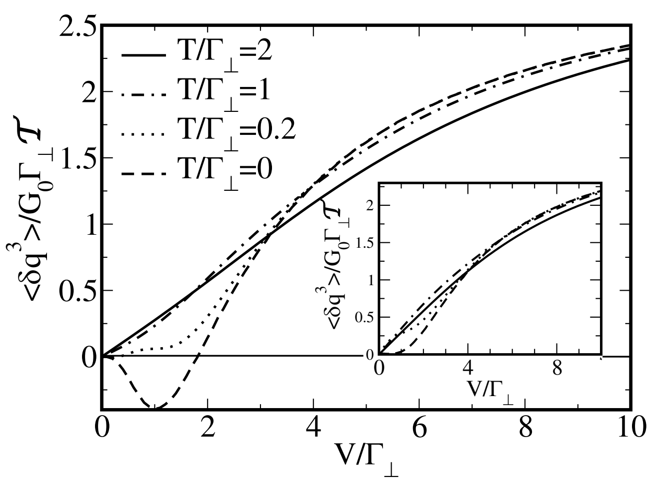

FIG. 1. The voltage dependence of the third cumulant for different temperatures and zero magnetic field ($\Delta = 0$) for $\Gamma_-/\Gamma_\perp = 0$ (main graph) and for $\Gamma_-/\Gamma_\perp = 0.9$ (inset).

comparison to $\langle \delta q^2 \rangle / \langle \delta q \rangle$, which is indeed in accordance with (9).

We now briefly turn to the $g = 1/2$ RT setup. This setup has caused much interest recently; see Ref. [22] and references therein. The Hamiltonian now is

$$H = H_0 + \gamma(\psi_L d^\dagger + d\psi_R^\dagger + \text{H.c.}) + \Delta d^\dagger d + H_C, \quad (13)$$

where H_0 stands for two biased LLs at $g = 1/2$, d is the electron operator on the dot, γ is the tunneling amplitude, and H_C is an electrostatic interaction we do not write explicitly here (see [22]). Introducing λ as standard, and carrying out the bosonization-refermionization analysis, we find the same set of equations as for the Kondo dot, Eqs. (5) and (6), but with $\lambda \to \lambda/2$ and $J_\perp = 2\gamma$, $J_\pm = 0$, when the Kondo statistics simplifies to binomial (unphysical case). Consequently, the FCS is given by a modification of the Levitov-Lesovik formula:

$$\chi_{1/2}(\lambda) = \chi_0^{1/2}(\lambda; 2V; \{T_\Delta(\omega)\}), \quad (14)$$

with the effective transmission coefficient $T_\Delta(\omega) = 4\gamma^4\omega^2/[4\gamma^4\omega^2 + (\omega^2 - \Delta^2)^2]$ of the RT setup in the symmetric case [22] (the contact asymmetry is unimportant). All the cumulants are thus obtainable from those of the noninteracting statistics Eq. (1).

The $\Delta = 0$ RT setup is equivalent to the model of direct tunneling between two $g = 2$ LLs [23]. The latter model is connected by the strong to weak coupling ($1/g \to g$) *duality* argument to the $g = 1/2$ Kane and Fisher model [24], which is, in turn, equivalent to the CB setup studied by KT. Therefore, their FCS must be related to our Eq. (14) at $\Delta = 0$ by means of the transformation: $T_0 \to 1 - T_0$ and $V \to V/2$. Indeed after some algebraic manipulation with KT's Eq. (12), we find that the FCS for the CB setup can be rewritten as:

$$\chi_{CB}(\lambda) = \chi_0^{1/2}(-\lambda; V; \{1 - T_0(\omega)\}).$$

To summarize, we derived the generating function for the charge transfer statistics for the Kondo dot in the Toulouse limit and analyzed the third cumulant in detail. At low temperatures the transport is accomplished by electrons as well as electron pairs in the generic case whereas at $T = 0$ the conventional binomial statistics is restored.

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