Inducing Chaos by Resonant Perturbations: Theory and Experiment

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We propose a scheme to induce chaos in nonlinear oscillators that either are by themselves incapable of exhibiting chaos or are far away from parameter regions of chaotic behaviors. Our idea is to make use of small, judiciously chosen perturbations in the form of weak periodic signals with time-varying frequency and phase, and to drive the system into a hierarchy of nonlinear resonant states and eventually into chaos. We demonstrate this method by using numerical examples and a laboratory experiment with a Duffing type of electronic circuit driven by a phase-locked loop. The phase-locked loop can track the instantaneous frequency and phase of the Duffing circuit and deliver resonant perturbations to generate robust chaos.

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It has been recognized that chaotic dynamics can be beneficial in natural systems and in practical applications as well. For instance, there are biological situations where inducing and maintaining chaotic motion are desirable [1], and previous works demonstrated that these can be achieved by applying small perturbations to an available parameter or dynamical variable of the system [1–3]. There are also practical applications where it is desirable to induce chaos in nonlinear oscillators operating in a stable regime, which could be far away from any transient chaotic behavior. For instance, if the goal is to disable a hostile device whose operation relies on stable dynamics of some embedded nonlinear oscillators (e.g., electronic circuits), inducing chaos in them can potentially ''confuse'' the device so that it will fail in its intended mission. For simple nonlinear oscillators such as the Duffing system, resonant perturbations can be used to drive the system in and out of chaotic motion [4,5], provided that the system equations are known so that the external excitation can be designed accordingly.

To our knowledge, in all existing works on inducing or maintaining chaos, either the system is in a transiently chaotic regime with some externally accessible parameter or state variable or the system equations are available [1–5]. For instance, in the experimental work on chaos maintenance by In *et al.* [3], it is required that the system exhibit intermittency between chaotic and periodic phases, i.e., that there be transient chaos, and that an accessible system parameter or variable be available. In another general theoretical work on chaos maintenance using weak harmonic perturbations [5], both system equations and transient chaos are required. *Our interest is in situations where the system can be far away from any chaos, transient or permanent, the system equations are not known, and no parameter or variable of the system can be accessed for direct adjustment.* Under the circumstance, a viable approach to disturb the system is to apply external excitations. In this Letter we present a general approach for inducing chaos by using small, resonant perturbations, and demonstrate its feasibility numerically and experimentally. In particular, we apply judiciously chosen perturbations to drive the oscillator into higher and higher resonant states. The perturbations can be a sinusoidal signal with time-varying frequency and phase, and how they vary is determined by a real-time measured signal emitted from the oscillator. Our goal is to control the perturbing field so as to maximize its effect on the system, thereby driving the system as far away from its equilibrium as possible, eventually generating chaotic dynamics. For experimental implementation, we use the principle of phase-locked loop (PLL), which is capable of continuously tracking the instantaneous frequency and phase of a target circuit and delivering proper resonant perturbations. We use an experimental Duffing circuit, which feeds a signal to a PLL, and demonstrate that an excitation of amplitude from the PLL of about 10% of the maximum circuit voltage oscillation can induce robust chaotic motion in the circuit. (Here ''robust'' means that when a system parameter is changed, such as the driving frequency, there are no periodic windows [6] amid the induced chaotic attractors.)

To gain insight, we imagine a simple, linear Hamiltonian system: the harmonic oscillator. Without any external perturbation, the system exhibits simple stable motion; it fundamentally prohibits any chaotic motion. Thus, showing that chaos can be induced in such a system by small resonant perturbations demonstrates the power of our method. The dynamics of the unperturbed Hamiltonian system is described by $d^2x/dt^2 = -dV(x)/dx$, where $V(x)$ is a potential function. In general, $V(x)$ can be any differentiable function. We assume, however, that V has a minimum and a maximum. Although our method works for any potential satisfying these constraints, we focus on the pendulum potential given by $V(x) = -\cos(x)$. The maxima at $x = \pm\pi$ define *hyperbolic orbits* at energy $E = 1$.

The hyperbolic orbits separate regions of confined and unconfined motion. Widespread chaos arises in the vicinity of the hyperbolic orbits due to homoclinic or heteroclinic intersections caused by arbitrarily small perturbations [7]. The oscillating frequency of the unperturbed system is a function of the energy: $\omega = \omega(E)$, where ω is defined within the region of the confined motion ($-1 \leq E \leq 1$). The frequency at the minimum is $\omega(-1) = 1$, and it decreases toward 0 as E approaches 1, because it takes an infinite amount of time for a trajectory to go from one hyperbolic orbit to another. We stress that this last feature is not particular of the pendulum potential, but it is true of all hyperbolic orbits: they have an infinitely long period. As the energy for hyperbolic orbit is approached, the period diverges, and the frequency goes to zero. This is important for our method.

Consider now that the system is set up with an initial energy $E_0 < 1$. In the absence of perturbations, it will keep oscillating with this constant energy. Our goal is to apply a small perturbation so that the energy is increased toward $E_{\max}$, where the homoclinic orbits lie, around which there is sustained chaos. The key observation is that the system's natural frequency changes with the energy. We must therefore change the frequency of the perturbation so that it always matches the natural frequency, thus ensuring that the resonant condition is satisfied at all times. The frequency of the external excitation thus changes with time, and we write $\nu(t)$. The form of $\nu(t)$ cannot be written down explicitly, because it is adjusted in response to the time variation of the natural frequency of the system. The equation of motion of the perturbed system is $d^2x/dt^2 = -dV(x)/dx + F\sin[\nu(t)t + \phi(t)]$, where $\phi(t)$ is a time-dependent phase. Although for the particular case of the pendulum potential, the frequency as a function of the energy is a known analytical function (expressed in terms of the elliptical functions), we want to keep our method as general as possible, and so we assume such a dependency is not known. In fact, we do not assume any knowledge of the potential, other than the fact that it has a minimum (with a region of confined, oscillating motion) and a hyperbolic orbit. In other words, we require only that the underlying system be *oscillatory*. Therefore a feasible way to determine the natural frequency of the system at a given time is through the observed dynamics, for instance, through the observation of the dynamical variable $x(t)$. We cannot measure the period directly from the dynamics, since the forcing term makes the motion aperiodic. However, since the perturbation is small ($F \ll 1$), at any given time the motion is *almost* periodic, meaning that the energy changes only very slowly with time. Typically, the system oscillates many times with only a small change in E and, hence, the resonant frequency changes very little as well. Using this fact, we define ν for a given time t as the average over the past Δn oscillations, where Δn is small enough so that the energy does not change appreciably in the corresponding time interval. In this way ν is defined purely in terms of the observed quantities of the system, and knowledge of the potential and/or the *equations of motion are not assumed.* The only requirement is that the average oscillating frequency of the system as a function of time be measured. In principle, the forcing term involves a time delay because the forcing is equivalent to a memory term. In our simulations, we consider only one oscillation in the past. The results we present are independent of Δn, to within the constraint mentioned.

Since we want the energy to increase in time so that the system approaches the hyperbolic orbit, we have to adjust the phase ϕ so that the forcing term is always in phase with the system's oscillation. We do so by making adjustments in discrete times: every time x crosses 0 in the positive direction, we change $\phi(t)$ so that the forcing term is in phase with $x(t)$. For a real circuit, this could be achieved continuously by a phase-locking scheme. Imposing this phase-adjusting mechanism, we ensure that *energy is always transferred from the perturbing force to the system, and not the other way around.* (Otherwise, the phase would drift in time, and the energy would not increase monotonically in time as we wish, but would instead oscillate more or less randomly.)

We now consider dissipative systems. For a given energy, call the average energy input rate due to the forcing κ_{in} and the energy output rate due to dissipation κ_{out}. Then, κ_{out} usually increases the farther the system is from the equilibrium point, which could be a stable fixed point or a stable cycle. The total energy of the system will stop increasing when κ_{out} equals κ_{in}. There are two possible scenarios: (1) If this happens for an energy *above* the energy of the hyperbolic orbit $E_{\max}$, we will be able to achieve the goal of exciting the system to near $E_{\max}$ and therefore inducing chaos. (2) If, however, κ_{out} becomes equal to κ_{in} for an energy E less than $E_{\max}$, the system will saturate at that energy, and we will not be able to push it to the neighborhood of the hyperbolic orbit. We can expect that, for a fixed forcing amplitude, as the dissipation increases from 0, a transition from case (1) to case (2) will occur. Thus, for a given amount of dissipation, the perturbation strength needs to be larger than a minimum value to ensure that the system can be driven to chaos. Another feature in dissipative systems is that, when chaos is induced, in order to maintain it, the frequency of the external excitation may need to be adjusted continuously to keep the energy of the system at about $E_{\max}$. The reason is that, when the system is driven to chaos, external energy is still needed to be delivered to the system to keep it in the chaotic state due to the dissipation.

We have tested our method using the following periodically forced Duffing's oscillator (in a dimensionless form):

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} - x + x^3 = A\sin(\nu_1 t + \phi_1), \tag{1}$$

where $\alpha = 0.05$, $A = 1$, $\nu_1 = 4500$, and $\phi_1 = 0$. For this parameter setting the system exhibits a period-1 attractor, and, hence, it can be regarded as being *far from chaos.* To induce chaos, we apply resonant perturbation of the form $F \sin[\nu_2(t)t + \phi_2(t)]$ to the right-hand side of Eq. (1), where $\nu_2(t)$ and $\phi_2(t)$ are estimated from the measured time series $x(t)$, as we have described. Figure 1(a) shows the evolution of the relative frequency $\nu_2(t)/\nu_1$ as chaos is being generated, together with a few representative phase-space plots of the attractor at different stages. We see that the frequency of the required resonant perturbation is decreased and remains approximately at a constant when chaos is induced, as predicted by our theory. Because of dissipation, the final frequency is finite. Figure 1(b) shows the evolution of the energy of the system: $E(t) = (\dot{x})^2/2 + x^2/2 - x^4/4$, relative to its maximum value $E_{\max} \approx 0.37$. The initial energy assumes a negative value, but it becomes approximately constant with small fluctuations after chaos sets in. Again, due to dissipation, the average final energy cannot reach its maximum possible value.

Experimentally, a convenient device that is capable of tracking both the frequency and the phase of a nonlinear oscillator is PLL [8,9]. To demonstrate the feasibility of our method, we have constructed a prototype system consisting of a Duffing type of circuit as the target oscillator to be driven into chaos and a PLL circuit that delivers resonant perturbation. An implementation of the Duffing circuit was proposed by Silva and Young [10], which is capable of chaos-based information processing in frequency up to 150 MHz. Our circuit is a low-frequency version of their circuit and is built using a quad-operational-amplifier TL084 chip and IN4004 diodes, as shown in Fig. 2. The circuit is powered by a voltage source of ± 9 V. The outputs are the voltage across the resistor R7 and that from the operational amplifier U2. The signals are digitally recorded using a digitizer (National Instrument) and analyzed using a LabView Virtual Instrument. In order to obtain the phase-space plot of the outputs for visualization, a differential amplifier was added to generate a voltage proportional to the potential across R7. The output signal from Y_1 is fed into a standard PLL circuit (LM565 integrated circuit), which tracks the frequency and phase of the output signal and provides resonant driving to the Duffing circuit through point X in Fig. 2.

The Duffing circuit is driven by a square-wave signal of amplitude 2 V and frequency 4 kHz to $X1$ in Fig. 2. To emphasize the point that the chaotic attractor can be induced by our method when the system is far away from chaos, we show in Fig. 3 an experimentally obtained bifurcation diagram, where the parameter value indicated by the vertical arrow is used for the experimental test. At this setting, the circuit exhibits a stable periodic attractor, as shown in Fig. 4(a), phase-space plot from two outputs of the circuit. The largest Lyapunov exponent is estimated to be $\lambda_1 \approx 0$ using a standard time series method [11]. There is apparently no transient chaos associated with the period-1 attractor. When the PLL delivers a resonant perturbation to the circuit, it goes into chaos, as desired. Figure 4(b)

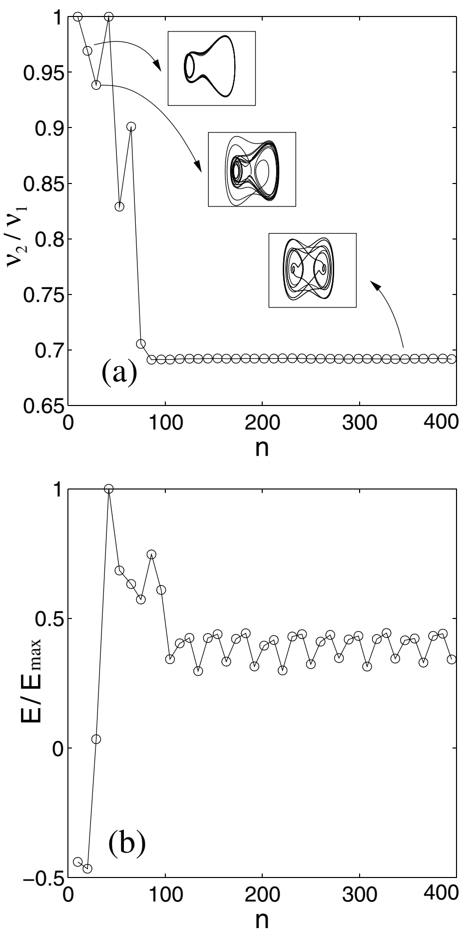

FIG. 1. For the periodically forced Duffing oscillator described by Eq. (1) that exhibits a period-1 attractor, (a) change in the frequency in the external resonant perturbation and three representative attractors [in the $(x, dx/dt)$ plane] during the induction of chaos. (b) Evolution of the energy of the oscillator. Here time n denotes the number of cycles of the periodic forcing.

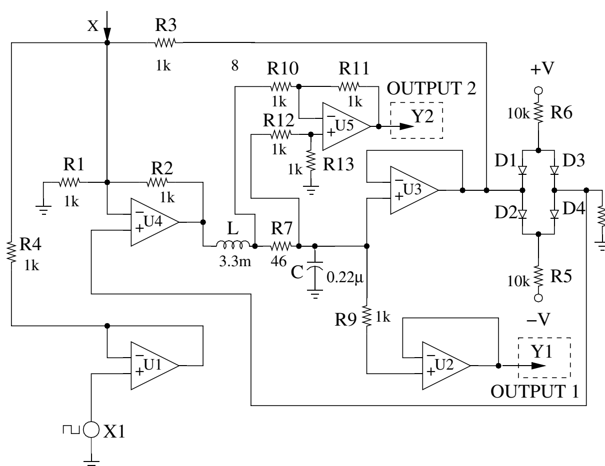

FIG. 2. Young-Silva circuit implementation of the Duffing oscillator. A differential amplifier (U5) was added to generate a single output proportional to the voltage across (R7) in order to visualize the phase plot of the voltage at Y1 versus that at Y2 on an oscilloscope.

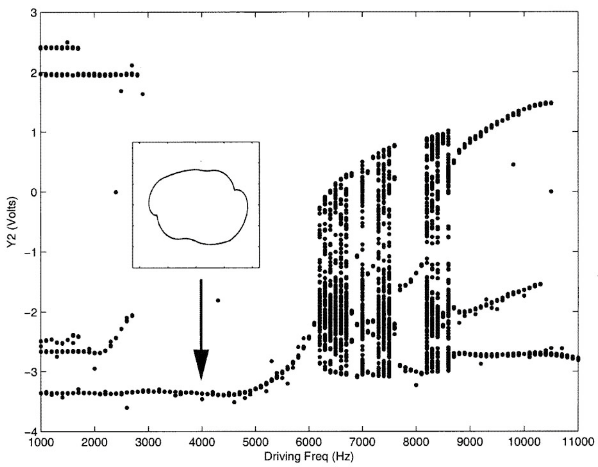

FIG. 3. Experimental bifurcation diagram for the Duffing circuit. The vertical arrow indicates the initial period-1 state of the system that is to be brought into chaos using a resonant perturbation.

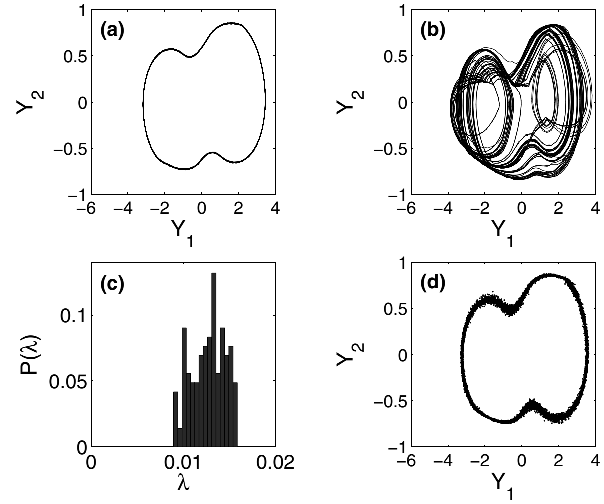

FIG. 4. (a) Phase-space plot of a periodic attractor from the Duffing circuit, (b) induced chaotic attractor under resonant driving, (c) histogram of the largest Lyapunov exponent from the induced chaotic attractor, and (d) attractor due to noisy driving, which is nonchaotic. The units of Y_1 and Y_2 in (a), (b), and (d) are in volts.

shows the phase-space plot of the Duffing circuit under resonant driving of the amplitude of 0.75 V from the PLL. The attractor is apparently chaotic. A histogram of the largest Lyapunov exponent estimated from the time series is shown in Fig. 4(c), the center of which is $\lambda_1 \gtrsim 0.01 > 0$, indicating that the attractor is indeed chaotic. To show the effectiveness of the resonant driving in inducing chaos, we replace the resonant driving by a noisy signal of larger amplitude (6 V) through a series resistor of 1 kΩ to point X. The resulting phase-space plot is shown in Fig. 4(d), which is only a smeared version of the periodic attractor in Fig. 4(a). The largest Lyapunov exponent is estimated to be $\lambda_1 \approx 0$.

In summary, we have demonstrated that a regular system far away from any complicated motion, can be driven to chaos through external, time-dependent, small resonant perturbations. We conceive the following situation of application: a signal is measured from the system that is to be driven to chaos and the instantaneous frequency and phase of the system are estimated using the signal, based on which continuous-time resonant perturbations are delivered to the system. As the frequency and phase of the system are changing, those of the perturbations are changed accordingly to maintain the resonant condition. As a result of the continuous resonant excitations, the system can be driven to and maintained in a chaotic state. The system can be, for example, an electronic circuit, and the external perturbations are from a microwave source whose frequency and phase can be continuously adjusted. We expect our method be useful for a variety of applications where chaos is desired.

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