Scattering Length Scaling Laws for Ultracold Three-Body Collisions

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We present a simple and unifying picture that provides the energy and scattering length dependence for all inelastic three-body collision rates in the ultracold regime for three-body systems with short-range two-body interactions. Here, we present the scaling laws for vibrational relaxation, three-body recombination, and collision-induced dissociation for systems that support s-wave two-body collisions. These systems include three identical bosons, two identical bosons, and two identical fermions. Our approach reproduces all previous results, predicts several others, and gives the general form of the scaling laws in all cases.

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The use of external magnetic fields to control the atomic interactions in trapped ultracold quantum gases has proven to be an extraordinary tool to explore different quantum regimes. At low temperatures, only the two-body s-wave scattering length a is needed to characterize the atomic interactions and can assume practically any value from $-\infty$ to $+\infty$ by tuning a magnetic field near a diatomic Feshbach resonance. This tunability has been used, for instance, to convert fermionic atoms into weakly bound bosonic molecules which, in some cases, were remarkably long lived [1]. In the quantum degenerate regime, this system provides a unique opportunity to explore the crossover between a Bose-Einstein condensate (BEC) of molecules and the BEC of atomic Cooper pairs, the BEC–BCS (Bardeen-Cooper-Schrieffer) transition.

Since the atomic and molecular lifetimes are influenced by three-body processes, these experiments underscore the importance of knowing the dependence of ultracold three-body collision rates on a. In particular, vibrational relaxation releases enough kinetic energy to free the collision products from typical traps, leading to molecular loss. Three-body recombination and collision-induced dissociation can also contribute to atomic and molecular loss, respectively. In ultracold collisions, the process that dominates is determined in large part by the threshold and scattering length scaling laws.

Threshold laws, which give the energy dependence for small collision energies, dictate the dominant partial wave for each process. The threshold law for elastic two-body collisions, for instance, gives an s-wave cross section that is constant at threshold while the lth partial wave is suppressed by a factor of E^{2l}. Combined with permutation symmetry requirements, the threshold law thus leads to the conclusion that the cross section for two identical fermions vanishes at low energies. Three-body threshold laws similarly depend on the number and kind of identical particles for each partial wave J^π, where J is the total orbital angular momentum and π is the parity [2].

While general results exist for three-body threshold laws, no similarly general scattering length scaling laws have yet been obtained. Results for some specific cases have been obtained using a variety of methods from simple to complex. The recombination rate for $B + B + B$ collisions, where B is a boson, for instance, scales roughly as a^4 for 0^+ and all a [3–5], and as a^8 for 2^+ and $a > 0$ [6]. For $F + F + F'$ collisions, where F and F' are fermions, recombination scales as a^6 for $a > 0$ [7]. It has also been shown for $a > 0$ that the relaxation rate for $BB + B$ collisions is linear in a [5], but is $a^{-3.33}$ for $FF' + F$ [8].

In this Letter we present a simple and unifying physical picture within which both the energy and scattering length dependence of *all* ultracold three-body collision rates (for short-range two-body interactions) can be derived and understood. Such a picture is possible because all systems can be represented by one of four prototype systems (two each for $a > 0$ and $a < 0$). In all cases, though, the rate limiting step is tunneling through a potential barrier in the initial channel (determined here from the adiabatic hyperspherical representation [9]). A simple WKB approximation to the tunneling probability is then sufficient to give both the energy and scattering length dependence.

The present approach reproduces the known results in a simple and conceptually clear manner and provides a general form for the scaling laws in all other cases, including arbitrary J^π. In addition, our results demonstrate that both threshold and scaling laws depend only on the initial collision channel and make explicit the pervasive influence of Efimov physics [10] on ultracold three-body collisions. We have restricted our discussion to systems with equal masses and to symmetries that support s-wave two-body collisions, representing most cases of experimental interest. We note that the present results apply only in the threshold regime, i.e., when the collision energy is the smallest energy in the system [6].

In the adiabatic hyperspherical representation, the three-body effective potentials and couplings are determined from the adiabatic equation [9],

$$H_{\rm ad}(R, \Omega)\Phi_\nu(R; \Omega) = U_\nu(R)\Phi_\nu(R; \Omega), \qquad (1)$$

where Ω denotes all hyperangles and R is the hyperradius that, roughly speaking, gives the overall size of the system. The adiabatic Hamiltonian $H_{\rm ad}$ includes the hyperangular

kinetic energy as well as all interactions. By expanding the total wave function on the adiabatic basis Φ_ν, the Schrödinger equation is reduced to (atomic units will be used unless otherwise noted)

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2}+W_\nu\right]F_\nu+\sum_{\nu'\neq\nu}V_{\nu\nu'}F_{\nu'}=EF_\nu. \quad (2)$$

In this expression, μ is the three-body reduced mass, E is the total energy, F_ν is the hyperradial wave function, $V_{\nu\nu'}$ is the nonadiabatic coupling responsible for inelastic transitions, and W_ν is the effective potential.

For short-range two-body interactions, the asymptotic behavior of W_ν can be derived analytically [11], yielding

$$W_\nu=E_{\upsilon l'}+\frac{l(l+1)}{2\mu R^2}, \qquad W_\nu=\frac{\lambda(\lambda+4)+15/4}{2\mu R^2}, \quad (3)$$

for the molecular and continuum channels, respectively. The molecular bound state energy $E_{\upsilon l'}$ is labeled by the rovibrational quantum numbers υ and l'; l is the atom-molecule relative angular momentum; and λ is a positive integer that labels the eigenstates of the kinetic energy [2].

In the limit $|a|\gg r_0$, where r_0 is the characteristic size of the two-body potential, Eq. (3) applies only for $R\gg|a|$. The modifications to W_ν for $r_0\ll R\ll|a|$ are directly responsible for the intriguing phenomenon known as the Efimov effect [10]. In this range, the potentials are still proportional to R^{-2}, but can now be attractive as well as repulsive. These modifications must be considered in order to properly predict the dependence of the three-body rates on a [3–6]. Strictly speaking, the term "Efimov effect" applies only to the emergence of an infinity of three-body bound states for $|a|\to\infty$. We will instead use the term "Efimov physics" to indicate the qualitative change in behavior exhibited by any system whenever at least two of the three possible scattering lengths are large.

Based on the modifications due to Efimov physics, we can classify all three-body systems into one of two categories: those with an attractive potential for $r_0\ll R\ll|a|$ and those without. For equal-mass systems, only 0^+ bosonic systems fall into the first category. In these systems, the attractive potential appears in the highest vibrationally excited s-wave molecular channel for $a>0$, and in the lowest continuum channel for $a<0$. The effective potentials for all higher channels are repulsive. These potentials are conveniently parametrized by

$$W_\nu(R)=-\frac{s_0^2+\frac{1}{4}}{2\mu R^2} \quad \text{and} \quad W_\nu(R)=\frac{s_\nu^2-\frac{1}{4}}{2\mu R^2}. \quad (4)$$

The coefficients s_0 and s_ν depend on the number of resonant pairs as well as the number of identical particles. For all other cases, including $J>0$, the relevant potentials are always repulsive. Like the 0^+ bosonic case, these potentials are parametrized by coefficients p_0 and p_ν:

$$W_\nu(R)=\frac{p_0^2-\frac{1}{4}}{2\mu R^2}, \qquad W_\nu(R)=\frac{p_\nu^2-\frac{1}{4}}{2\mu R^2}. \quad (5)$$

In all cases, deeply bound molecular channels are essentially independent of a. The coefficients in Eqs. (4) and (5) can be obtained analytically [10], and numerical values are shown in Table I for the two most dominant partial waves for relaxation and recombination near threshold. Table I also shows the minimum l and λ [see Eq. (3)] allowed by permutation symmetry for each partial wave [2].

The relevant potentials and couplings are sketched in Figs. 1(a) and 1(b) for 0^+ bosonic systems and in Figs. 1(c) and 1(d) for all other cases. For $r_0<R<|a|$, the potentials are given by Eqs. (4) and (5); for $R>|a|$, the potentials are those in Eq. (3). The lowest continuum channel, labeled "α," is the initial channel for recombination. Channel "β" is the weakly bound molecular channel and is the initial channel for relaxation, while "γ" is a deeply bound molecular channel. The nonadiabatic couplings in Fig. 1 indicate the regions where inelastic transitions are most likely. From our numerical calculations and on physical grounds, we believe that Fig. 1 represents all three-body systems near threshold.

Knowing the dependence of the potentials and couplings on R and a enables us to derive the scaling laws from the definitions of the rates,

$$V_{\rm rel}\propto|T_{fi}|^2/k, \qquad K_3\propto|T_{fi}|^2/k^4, \quad (6)$$

in terms of the transition probabilities $|T_{fi}|^2$. Only the wave vector important for the threshold law has been included—$k^2=2\mu(E-E_{\upsilon l'})$ for $V_{\rm rel}$ and $k^2=2\mu E$ for K_3.

The transitions proceed via tunneling in the initial potential to the R where the coupling peaks and the transition probability can be approximated by the WKB tunneling probability (including the Langer correction [4]),

$$P_{x\to y}^{(\nu)}\approx\exp\left[-2\int_y^x\sqrt{2\mu\left(W_\nu(R)+\frac{1/4}{2\mu R^2}-E\right)}dR\right]. \quad (7)$$

The Langer correction is crucial for obtaining the correct

TABLE I. Coefficients of the potentials in Eqs. (3)–(5). Except for 0^+ bosons, the coefficients correspond to p_0 and p_ν.

	J^π	$l_{\rm min}$	$\lambda_{\rm min}$	$s_0(p_0)$	$s_\nu(p_\nu)$
BBB	0^+	0	0	1.006 237 8	4.465 294 6
	1^-	1	3	2.863 799 4	6.462 204 4
	2^+	2	2	2.823 341 9	5.508 249 4
BBB'	0^+	0	0	0.413 697 3	3.450 989 1
	1^-	1	1	2.278 741 3	3.641 303 5
FFF'	0^+	0	2	2.166 222 0	5.127 352 1
	1^-	1	1	1.772 724 3	4.358 249 3
	2^+	2	2	3.104 976 9	4.795 405 4

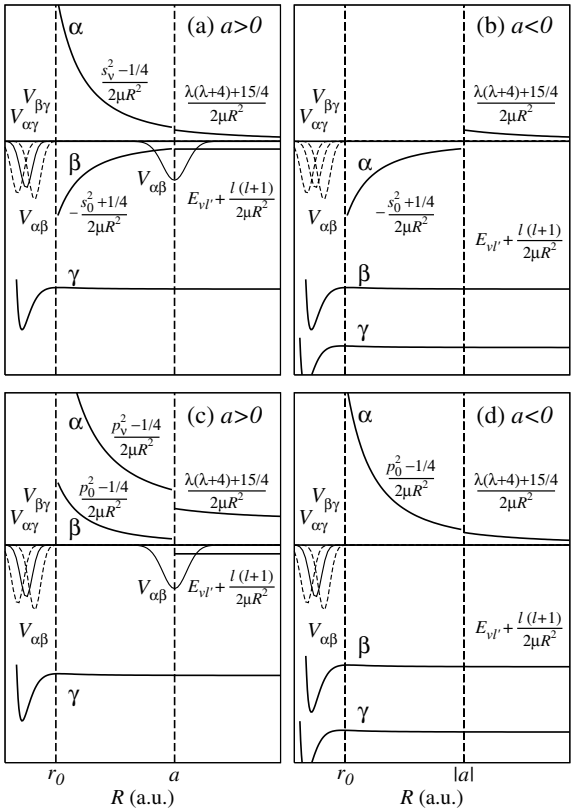

FIG. 1. Schematic plot of the three-body effective potentials for 0^+ bosonic systems with (a) $a > 0$ and (b) $a < 0$, and for all other systems with (c) $a > 0$ and (d) $a < 0$.

scaling with E and a. At ultracold temperatures, the classical turning point is much larger than $|a|$.

In all cases, the couplings peak at $R \approx |a|$, $R \approx r_0$, or both. Relaxation for $a > 0$, for instance, only occurs at small distances, $R \approx r_0$, where the coupling peaks [$V_{\beta\gamma}$ in Figs. 1(a) and 1(c)]. It follows that the tunneling probability in the initial channel β must be evaluated by integrating Eq. (7) from the classical turning point r_c to r_0. This range, however, spans both kinds of potentials so that the tunneling in the two regions must be included,

$$|T_{\gamma\beta}|^2 \simeq P^{(\beta)}_{r_c\to a} P^{(\beta)}_{a\to r_0}. \tag{8}$$

Similarly, the transition probability for recombination must also include tunneling in two different regions between r_c and r_0. Recombination for $a > 0$ can occur, though, at two distances—$R \approx a$ and $R \approx r_0$—since the coupling peaks both places, leading to

$$|T_{\beta\alpha}|^2 \simeq P^{(\alpha)}_{r_c\to a} + P^{(\alpha)}_{r_c\to a} P^{(\beta)}_{a\to r_0} + P^{(\alpha)}_{r_c\to a} P^{(\alpha)}_{a\to r_0}, \tag{9}$$

where each term corresponds to a different reaction pathway leading to recombination to the highest vibrationally excited state. Recombination to deeper bound states can easily be included and reproduces the results in Ref. [5]. Although not indicated here, these paths can interfere. The relative importance of each pathway, and of the interference between them, will be discussed below. For $a < 0$, all couplings peak at $R \approx r_0$. Applying the arguments above, relaxation and recombination are, respectively,

$$|T_{\gamma\beta}|^2 \simeq P^{(\beta)}_{r_c\to r_0}, \qquad |T_{\beta\alpha}|^2 \simeq P^{(\alpha)}_{r_c\to|a|} P^{(\alpha)}_{|a|\to r_0}. \tag{10}$$

We can now determine both the threshold laws and the scattering length scaling. Since W_ν has the same form in the region $R > |a|$ in all cases [Eq. (3)], the tunneling probabilities in this region are also the same,

$$P^{(\alpha)}_{r_c\to|a|} \propto (ka)^{2\lambda+4}, \qquad P^{(\beta)}_{r_c\to|a|} \propto (ka)^{2l+1}. \tag{11}$$

Equation (11) completely determines the threshold laws for each process [2]. The scaling with a, however, will be strongly modified by the tunneling probability in the region $r_0 < R < |a|$ due to Efimov physics.

For 0^+ bosons with $a > 0$, the β-channel potential is attractive in the region $r_0 < R < a$. Consequently, the probability $P^{(\beta)}_{a\to r_0}$ [Eqs. (8) and (9)] is not a tunneling probability, but rather a transmission probability that can be determined from general arguments based on the known solutions for Eq. (2) [5,10]. This analysis gives

$$V_{\rm rel} \propto \left[\frac{\sinh(2\eta)}{\sin^2[s_0\ln(a/r_0)+\Phi]+\sinh^2(\eta)}\right]a, \tag{12}$$

since $l = 0$ in Eq. (11). The constant η is related to the probability for transitions at small distances [5] and Φ is an unknown small-R phase. Equation (12) has been deduced [5] as a generalization of the result presented in Ref. [10]. In both cases, however, the linear dependence on a was obtained indirectly by dimensional and physical arguments, while it follows naturally from Eq. (11) in the present analysis.

For 0^+ bosons with $a < 0$, the initial state for relaxation is a deeply bound vibrational state independent of a. The transmission probability, and thus the relaxation rate, does not depend on a either, leading to

$$V_{\rm rel} \propto A_\eta k^{2l} r_0^{2l+1}. \tag{13}$$

Here and below, the constant A_η generically represents the small-R transition probability and can display resonant effects due to a three-body Feshbach resonance, but is otherwise independent of a.

For all cases other than 0^+ bosons, the initial channel for relaxation is repulsive for $r_0 < R < |a|$ [Figs. 1(c) and 1(d)] so that relaxation proceeds by tunneling only. For $a > 0$, the relaxation rate is

$$V_{\rm rel} \propto A_\eta k^{2l}\left(\frac{r_0}{a}\right)^{2p_0} a^{2l+1}, \tag{14}$$

which decreases with a whenever $2l + 1 < 2p_0$, yielding, in turn, longer molecular lifetimes. Relaxation for $FF' + F$ collisions, for instance, scales as $a^{-3.332\,444}$ (see Table I). This result agrees with the recent prediction of Petrov *et al.*

TABLE II. Threshold and scattering length scaling laws for three-body rates. Boldface indicates dominant contributions.

	J^π	$V_{\rm rel}$			$K_3(D_3)$		
		E	$a>0$	$a<0$	E	$a>0$	$a<0$
BBB	0^+	$\boldsymbol{const}$	$\boldsymbol{a}^{a}$	$\boldsymbol{const}$	$\boldsymbol{const(k^4)}$	$\boldsymbol{a^{4a}}$	$\boldsymbol{\lvert a\rvert^4}$
	1^-	k^2	$a^{-2.728}$	const	$k^6(k^{10})$	a^{10}	$\lvert a\rvert^{4.272}$
	2^+	k^4	$a^{-0.647}$	const	$k^4(k^8)$	a^{8b}	$\lvert a\rvert^{2.353}$
BBB'	0^+	$\boldsymbol{const}$	$\boldsymbol{a}$	$\boldsymbol{const}$	$\boldsymbol{const(k^4)}$	$\boldsymbol{a^4}$	$\boldsymbol{\lvert a\rvert^4}$
	1^-	k^2	$a^{-1.558}$	const	$k^2(k^6)$	a^6	$\lvert a\rvert^{1.443}$
FFF'	0^+	$\boldsymbol{const}$	$\boldsymbol{a^{-3.332c}}$	$\boldsymbol{const}$	$k^4(k^8)$	a^8	$\lvert a\rvert^{3.668}$
	1^-	k^2	$a^{-0.546}$	const	$\boldsymbol{k^2(k^6)}$	$\boldsymbol{a^{6d}}$	$\boldsymbol{\lvert a\rvert^{2.455}}$
	2^+	k^4	$a^{-1.210}$	const	$k^4(k^8)$	a^8	$\lvert a\rvert^{1.790}$

[a]Refs. [3–5].
[b]Ref. [6].
[c]Ref. [8].
[d]Ref. [7].

[8] and is consistent with experiments [1]. Like the 0^+ boson systems, relaxation does not depend on a for $a<0$,

$$V_{\rm rel} \propto A_\eta k^{2l} r_0^{2l+1}. \tag{15}$$

The present analysis applies equally well to recombination. For instance, recombination of 0^+ bosons with $a>0$ is determined from Eq. (9), which includes three different recombination paths. Only the interference between the first two terms in Eq. (9) will be included here, since we expect the third term will be suppressed. In fact, this interference is well known [3–5], and the present analysis reproduces the known expression plus a modification:

$$K_3 \propto \left[\sin^2[s_0\ln(a/r_0)+\Phi] + A_\eta\left(\frac{r_0}{a}\right)^{2s_\nu}\right]a^4. \tag{16}$$

The first term in Eq. (16) is the usual result, while the second is due to recombination at small R and was not predicted in Refs. [3–5]. Because the small-R coupling lies so far into the classically forbidden region, we expect A_η to be small. For $a<0$, the present analysis yields the same expression found in Ref. [5],

$$K_3 \propto \left[\frac{\sinh(2\eta)}{\sin^2[s_0\ln(|a|/r_0)+\Phi]+\sinh^2(\eta)}\right]a^4. \tag{17}$$

The recombination rate for all other cases can be determined, giving for $a>0$ and $a<0$, respectively,

$$K_3 \propto k^{2\lambda}\left[1 + A_\eta\left(\frac{r_0}{a}\right)^{2p_0} + B_\eta\left(\frac{r_0}{a}\right)^{2p_\nu}\right]a^{2\lambda+4} \tag{18}$$

$$K_3 \propto k^{2\lambda}\left(\frac{r_0}{|a|}\right)^{2p_0}|a|^{2\lambda+4}, \tag{19}$$

predicting an asymmetry in K_3 for $a<0$ and $a>0$. The constants A_η and B_η are expected to be small.

We have also solved Eq. (2) numerically using model two-body potentials, confirming Eqs. (12)–(15) for $BB+B$ and $FF'+F$ collisions. For $FF'+F$, contributions for both s-wave and p-wave final states scale as $a^{-3.33}$, emphasizing that the scaling only depends on the initial state.

Table II summarizes the scaling laws, showing only the main power-law behavior of each rate. The two dominant partial waves, determined by their energy dependence, are shown for each process. We also have indicated the previously known results as well as the competition from the next leading term for finite temperatures. For fermion relaxation, this term is comparatively more important than for bosons. For completeness, we include the results for the dissociation rate D_3 (for which $k^2=2\mu E$).

In this Letter, we have deduced the scaling laws for ultracold three-body collision rates for *all* equal-mass three-body systems and symmetries that support s-wave two-body collisions. We have developed a simple and unifying approach that describes intuitively all three-body collision processes in the same framework. Our analysis shows that for relaxation and recombination, the scaling laws as well as the threshold laws depend only on the initial state. Any dependence on the final state is expected to enter via the coupling terms, and should at most be weakly dependent on energy and scattering length. These results demonstrate yet again the remarkable influence of Efimov physics in ultracold three-body collisions, even for systems without bosons.

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