Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential

J. C. Osborn,[1] K. Splittorff,[2] and J. J. M. Verbaarschot[3]
[1]Physics Department, Boston University, Boston, Massachusetts 02215, USA
[2]Nordita, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark
[3]Department of Physics and Astronomy, SUNY, Stony Brook, New York 11794, USA
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The relation between the spectral density of the QCD Dirac operator at nonzero baryon chemical potential and the chiral condensate is investigated. We use the analytical result for the eigenvalue density in the microscopic regime which shows oscillations with a period that scales as $1/V$ and an amplitude that diverges exponentially with the volume $V = L^4$. We find that the discontinuity of the chiral condensate is due to the whole oscillating region rather than to an accumulation of eigenvalues at the origin. These results also extend beyond the microscopic regime to chemical potentials $\mu \sim 1/L$.

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Introduction.—One of the salient features of QCD at low energy is the spontaneous breaking of chiral symmetry characterized by a discontinuity of the chiral condensate. More than two decades ago it was realized by Banks and Casher [1] that the discontinuity of the chiral condensate at zero quark mass is proportional to the eigenvalue density of the QCD Dirac operator. This relation establishes that the eigenvalue spectrum of the anti-Hermitian Dirac operator becomes dense at the origin of the imaginary axis in the thermodynamic limit. The Banks-Casher relation is of substantial practical value for nonperturbative numerical studies of QCD. It allows one to extract the chiral condensate directly from the spectral density.

At nonzero baryon chemical potential, μ, the Euclidean Dirac operator, $D \equiv D_\eta \gamma_\eta + \mu\gamma_0$, is non-Hermitian so that the support of its spectrum is a two-dimensional domain in the complex plane. In this case we will show that the discontinuity of the chiral condensate at zero mass is *not* due to the accumulation of eigenvalues near zero.

The QCD partition function at zero temperature does not depend on the baryon chemical potential,

$$Z_{N_f}(m;\mu) = Z_{N_f}(m;\mu = 0), \quad \text{for } \mu < \mu_c, \tag{1}$$

where μ_c is the smallest mass per unit quark number in the excitation spectrum. Here and below we only consider the case with N_f quark flavors with equal mass m. Therefore, the chiral condensate given by

$$\Sigma_{N_f}(m) = \frac{1}{N_f V}\partial_m \log Z_{N_f}(m;\mu), \tag{2}$$

remains equal to its value at $\mu = 0$ for $\mu < \mu_c$. Our aim is to understand this behavior from the spectrum of the Dirac operator.

We will consider two types of gauge field averages: quenched averages where the determinant of the Dirac operator is not included in the average and unquenched averages which include the fermion determinant. The quenched spectral density can be expressed as

$$\rho_Q(x, y;\mu) = \left\langle \sum_k \delta^2(x + iy - z_k) \right\rangle, \tag{3}$$

where the eigenvalues of the Dirac operator are given by z_k, and the brackets denote the average over the Yang-Mills action. The eigenvalue density of full QCD includes the fermion determinant in the average

$$\rho_{N_f}(x, y, m;\mu) = \frac{\langle \sum_k \delta^2(x + iy - z_k) \det^{N_f}(D + m)\rangle}{\langle \det^{N_f}(D + m)\rangle}. \tag{4}$$

Strictly speaking, since ρ_{N_f} is in general complex, this is not a density. Because of chiral symmetry the eigenvalues occur in pairs $\pm z_k$ so that $\rho_{N_f}(x, y, m;\mu) = \rho_{N_f}(-x, -y, m;\mu)$. A second reflection symmetry which holds only after averaging over the gauge field configurations is that $\rho^*_{N_f}(x, y, m;\mu) = \rho_{N_f}(x, -y, m;\mu)$. Because the fermion determinant vanishes for $z_k = \pm m$ we expect that $\rho_{N_f}(x = \pm m, y = 0, m;\mu) = 0$. The chiral condensate in the chiral limit can be expressed as

$$\Sigma = \lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\int dxdy \frac{\rho_{N_f}(x, y, m;\mu)}{x + iy + m}. \tag{5}$$

At zero chemical potential the above quantity is known to be proportional to the density of the imaginary eigenvalues at zero [1]. This happens because then $\rho_{N_f}(x, y, m;\mu = 0) \propto \delta(x)$. When $\mu \neq 0$ the eigenvalues spread into the complex plane and this argument no longer holds. In this case we will show that there is an extended region of the eigenvalue density that contributes to the chiral condensate.

For simplicity, here we will show how the condensate arises from the microscopic limit of the spectral density which is believed to be universal. We have also verified [2] that a similar mechanism occurs even for larger μ provided that the contributing eigenvalues are still in the universal region. The microscopic limit of the spectral density is defined as [3]

$$\hat{\rho}_{N_f}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \lim_{V\to\infty} \frac{1}{(\Sigma V)^2} \rho_{N_f}\left(\frac{\hat{x}}{\Sigma V}, \frac{\hat{y}}{\Sigma V}, \frac{\hat{m}}{\Sigma V}; \frac{\hat{\mu}}{F_\pi \sqrt{V}}\right). \tag{6}$$

In this limit, $\hat{x} = x\Sigma V$, $\hat{y} = y\Sigma V$, $\hat{m} = m\Sigma V$, and $\hat{\mu} = \mu F_\pi \sqrt{V}$ are kept fixed with Σ given by (5) and F_π the pion decay constant. The expression for the condensate (5) in this limit becomes

$$\Sigma = \lim_{\hat{m},\hat{\mu}\to\infty} \Sigma \int d\hat{x} d\hat{y} \frac{\hat{\rho}_{N_f}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu})}{\hat{x} + i\hat{y} + \hat{m}}. \tag{7}$$

The microscopic limit of the spectral density at nonzero chemical potential was recently calculated both for the quenched case [4] and the unquenched case [5]. For a nonzero number of flavors it was found [6] that the eigenvalue density for $m\Sigma < 2\mu^2 F_\pi^2$ is a strongly oscillating complex function. The oscillations cover a region of the complex eigenvalue plane and, as we will see below, the entire region contributes to the integral in (7). This constitutes a new mechanism where a discontinuity of the chiral condensate in the complex mass plane is obtained from an oscillating eigenvalue density in the complex plane. This mechanism does not rely on the specific form of the eigenvalue density as is demonstrated in the simple example below.

The lack of Hermiticity properties of the Dirac operator at nonzero chemical potential is a direct consequence of the imbalance between quarks and antiquarks imposed in order to induce a nonzero baryon density. Because of the phase of the fermion determinant, probabilistic methods are no longer effective in the analysis of the partition function. This is known as the sign problem. Although progress has been made in some areas we believe that because of its physical origin, a paradigm shift will be necessary to develop viable probabilistic algorithms for this problem. Because of this, it is our opinion that it is particularly important to improve our analytical understanding of chiral symmetry breaking for QCD at nonzero baryon density.

Euclidean QCD at finite baryon density is not the only system without Hermiticity properties that has received much attention recently. We mention the distribution of the poles of S matrices which are given by the eigenvalues of a non-Hermitian operator [7,8], the Hatano-Nelson model [9] (a random potential together with a nonzero imaginary vector potential), and the description of Laplacian growth in terms of the spectrum of non-Hermitian random matrices [10]. The essential difference from QCD is that in these problems the determinant of the operator only enters in the generating function of the resolvent. We will see that the additional determinant in QCD completely changes the character of the theory.

Example.—As an example to illustrate our point, let us consider the eigenvalue density (sgn is the sign function)

$$\hat{\rho}_{\rm Ex}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \frac{\theta(2\hat{\mu}^2 - |\hat{x}|)}{4\pi\hat{\mu}^2}[1 - e^{i\,{\rm sgn}(\hat{x})(|\hat{x}|+2\hat{\mu}^2)\hat{y}/4\hat{\mu}^2 + (|\hat{x}|-|\hat{m}|)(|\hat{x}|+2\hat{\mu}^2)/4\hat{\mu}^2}]. \tag{8}$$

This eigenvalue density has the same reflection symmetries as the eigenvalue density of the QCD Dirac operator and has the property that it vanishes at the point where the fermion determinant is zero. The integral in (7) can be evaluated analytically by means of a complex contour integral in $\hat{y}$ resulting in

$$\Sigma_{\rm Ex} = {\rm sgn}(\hat{m})\Sigma + \frac{\Sigma}{\hat{m}} e^{-|\hat{m}|}(e^{-|\hat{m}|} - 1), \tag{9}$$

which, for large $\hat{m}$, approaches ${\rm sgn}(\hat{m})\Sigma$. What we have learned from this example is that a discontinuity in the chiral condensate can be obtained from an oscillating spectral density rather than from eigenvalues localized on the imaginary axis. We will show next that the same mechanism is at work for QCD at $\mu \neq 0$.

The microscopic spectral density.—The input for our calculation of the chiral condensate is the microscopic spectral density derived for any number of flavors in [5]. Without loss of generality we will consider the case $N_f = 1$ and topological charge equal to zero for which the equations are less extensive. The microscopic eigenvalue density can be decomposed as [5,6]

$$\hat{\rho}_{N_f=1}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) - \hat{\rho}_U(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}), \tag{10}$$

with ($\hat{z} = \hat{x} + i\hat{y}$)

$$\hat{\rho}_U(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} e^{-(\hat{z}^2+\hat{z}^{*2})/(8\hat{\mu}^2)} K_0\left(\frac{|\hat{z}|^2}{4\hat{\mu}^2}\right) \frac{I_0(\hat{z})}{I_0(\hat{m})} \times \int_0^1 dt t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m} t). \tag{11}$$

The first term in (10) is the quenched eigenvalue density [4] given by $\hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) = \hat{\rho}_U(\hat{x}, \hat{y}, \hat{x} + i\hat{y}; \hat{\mu})$. As expected, the microscopic spectral density vanishes at $\hat{z} = \pm\hat{m}$. A plot of the real part of the eigenvalue density for $\hat{m} = 60$ and $\hat{\mu} = 8$ is shown in Fig. 1. Notice that the oscillatory region extends from the mass pole at $\hat{z} = \hat{m}$ and toward the boundary of the support of the spectrum.

The oscillations appear as the microscopic variables become large, i.e., as the thermodynamic limit is approached. In this region an asymptotic formula for the eigenvalue density is accurate and will be used in order to analyze the role of the oscillations for chiral symmetry breaking. We first derive the asymptotic formula for the eigenvalue density and then evaluate the chiral condensate from (7). For $\hat{\mu}^2 \gg 1$ and $(\hat{x} + \hat{m})/(4\hat{\mu}^2) < 1$, the integral in (11) over t is very well approximated by [11]

$$\int_0^1 dt t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m} t) \approx \frac{1}{4\hat{\mu}^2} \exp\left(\frac{\hat{z}^{*2} + \hat{m}^2}{8\hat{\mu}^2}\right) I_0\left(\frac{\hat{m}\hat{z}^*}{4\hat{\mu}^2}\right). \tag{12}$$

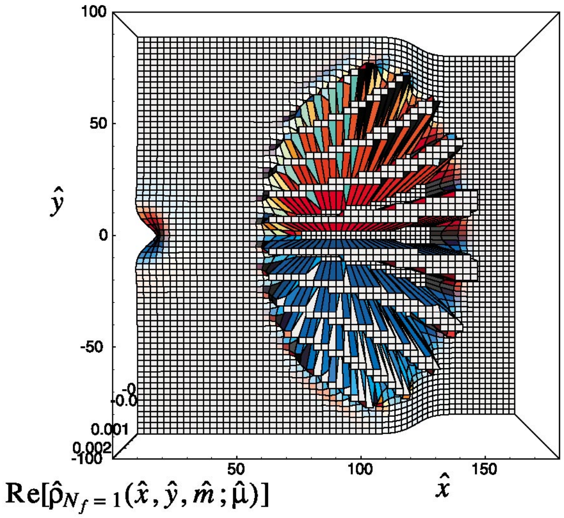

FIG. 1 (color). Top view of the real part of the eigenvalue density for one flavor with $\hat{m} = 60$ and $\hat{\mu} = 8$ in half of the complex eigenvalue plane. The oscillations are cut off to illustrate the eigenvalue repulsion at $\hat{x} = \hat{y} = 0$ and the dropoff for $\hat{x} > 2\hat{\mu}^2$.

Furthermore, we are interested in the approach to the thermodynamic limit where $|\hat{m}| \gg 1$, $|\hat{z}| \gg 1$, $|\hat{z}|^2/(4\hat{\mu}^2) \gg 1$, and $|\hat{m}\,\hat{z}|/(4\hat{\mu}^2) \gg 1$. This justifies the replacement of the Bessel functions by their leading order asymptotic expansion including the Stokes terms. We obtain the following asymptotic result for the difference between the quenched and the unquenched eigenvalue density

$$\hat{\rho}_U(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) \sim \frac{1}{4\pi\hat{\mu}^2} e^{-[\hat{y}+i\mathrm{sgn}(\hat{x})(|\hat{x}|+|\hat{m}|-4\hat{\mu}^2)]^2/(8\hat{\mu}^2)-(|\hat{x}|-2\hat{\mu}^2)^2/(2\hat{\mu}^2)}. \tag{13}$$

This expression has the reflection symmetries discussed below (4). The asymptotic expansion of the quenched part of the spectral density is simply given by

$$\hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) \sim \frac{1}{4\pi\hat{\mu}^2}\theta(2\hat{\mu}^2 - |\hat{x}|). \tag{14}$$

For the argument presented below it is important that also the asymptotic expansion of the spectral density vanishes at $\hat{x} + i\hat{y} = \pm\hat{m}$.

The chiral condensate.—As explained in the introduction, the chiral condensate does not depend on the baryon chemical potential. Hence, in the microscopic limit it is known that for zero topological charge [12,13] (momentarily we use the original variables to emphasize the volume dependence)

$$\Sigma_{N_f=1}(m) = \Sigma\frac{I_1(mV\Sigma)}{I_0(mV\Sigma)}, \tag{15}$$

which is discontinuous, $\Sigma_{N_f=1}(m) = \mathrm{sgn}(m)\Sigma$, in the thermodynamic limit at fixed quark mass. The question we wish to answer is how the oscillatory spectral density conspires into a μ independent condensate. We stress that this is not just a challenging mathematical problem; understanding which parts of the eigenvalue density contribute to the chiral condensate will give direct insight in the physical consequences of the sign problem.

We now derive the chiral condensate from (7) using the asymptotic form of the microscopic eigenvalue density. We first consider the integral over $\hat{y}$. The contribution from the quenched part of the spectral density (14) is given by

$$\frac{1}{4\pi\hat{\mu}^2}\int_{-\infty}^{\infty} d\hat{y}\frac{1}{\hat{x} + i\hat{y} + \hat{m}} = \mathrm{sgn}\,(\hat{x} + \hat{m})\frac{1}{4\hat{\mu}^2}. \tag{16}$$

The contribution from $\hat{\rho}_U$ in (7) is evaluated by a saddle point approximation. The contour in the complex $\hat{y}$ plane is deformed into a contour from $-\infty$ to ∞ over the saddle point at $\hat{y} = i\mathrm{sgn}(\hat{x})(4\hat{\mu}^2 - |\hat{x}| - |\hat{m}|)$ and, if the contour has crossed the pole, an integral around the pole at $\hat{y} = i(\hat{x} + \hat{m})$. The saddle point contribution is exponentially suppressed for $|\hat{x}| < 2\hat{\mu}^2$ leaving only the integral around the pole. For $\hat{m} > 0$ ($\hat{m} < 0$) the pole contribution for $\hat{x} > 0$ ($\hat{x} < 0$) is exponentially suppressed. We obtain

$$\frac{1}{4\pi\hat{\mu}^2}\int_{-\infty}^{\infty} d\hat{y} e^{-[\hat{y}+i\mathrm{sgn}(\hat{x})(|\hat{x}|+|\hat{m}|-4\hat{\mu}^2)]^2/8\hat{\mu}^2-(|\hat{x}|-2\hat{\mu}^2)^2/2\hat{\mu}^2}\frac{1}{\hat{x} + i\hat{y} + \hat{m}} \simeq -\frac{1}{2\hat{\mu}^2}[\theta(\hat{m})\theta(-\hat{x} - \hat{m}) - \theta(-\hat{m})\theta(\hat{x} + \hat{m})], \tag{17}$$

where we have used that the exponent vanishes at the pole. For $\hat{x} > 2\hat{\mu}^2$ the eigenvalue density is zero so it is now trivial to do the integration over $\hat{x}$ to get

$$\Sigma_{N_f=1} = \frac{\Sigma}{2\hat{\mu}^2}\int_{-2\hat{\mu}^2}^{2\hat{\mu}^2} d\hat{x}\left[\frac{1}{2}\mathrm{sgn}(\hat{x} + \hat{m}) + \theta(\hat{m})\theta(-\hat{x} - \hat{m}) - \theta(-\hat{m})\theta(\hat{x} + \hat{m})\right] = \mathrm{sgn}(\hat{m})\Sigma. \tag{18}$$

This result agrees with (15) for $|\hat{m}| \gg 1$ where the asymptotic expansion of $\Sigma_{N_f=1}(m)$ is valid. Using the exact microscopic spectral density we would have recovered the mass dependence of (15).

We also emphasize that a finite result for the chiral condensate is not obtained due to a cancellation of the pole and a zero of the fermion determinant. The pole term gives a finite contribution for each of the two terms in (10) which do *not* vanish at $\hat{z} = \hat{m}$. We have checked numerically that the same mechanism results in a discontinuity of the chiral condensate for more than one flavor.

The contribution from the unquenched part of the eigenvalue density to the chiral condensate is dominated by the pole term because the exponential in (17) suppresses the integrand at the saddle point in the complex $\hat{y}$ plane. The simple result (17) which implies a nonzero chiral condensate in the chiral limit is thus directly related to the complex phase of the spectral density. The oscillating exponential has to suppress terms that diverge exponentially with the volume which is achieved by oscillations in the $\hat{y}$ direction with a period that scales as the inverse of the volume. In contrast, for quenched QCD, where the eigenvalue density is real and positive, the chiral condensate vanishes in the chiral limit provided that $\mu \neq 0$ as follows immediately from (16). Instead, a diquark condensate forms and breaks chiral symmetry just like in phase quenched theories (see, e.g., [14,15]). Chiral symmetry breaking in phase quenched and full QCD occur by means of two different mechanisms, and the oscillations of the eigenvalue density due to the sign problem in the unquenched case distinguish between the two.

The resolvent.—One can easily convince oneself that for $\hat{z} \neq \hat{m}$ the exponent in the integrand of the resolvent

$$\hat{G}_{N_f}(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu}) = \Sigma \int d^2\hat{u} \frac{\hat{\rho}_{N_f}(\mathrm{Re}\,(\hat{u}), \mathrm{Im}\,(\hat{u}), \hat{m}; \hat{\mu})}{\hat{u} + \hat{z}} \tag{19}$$

results in expressions that diverge in the thermodynamic limit. Indeed, this was concluded from earlier random matrix calculations of the resolvent [16]. Using the microscopic spectral density (11) one finds good numerical agreement with the results obtained in [16].

Phase transitions in generating functions.—Our results can be understood in terms of phase transitions for generating functions for the spectral density. Using the replica trick [6,14,17]

$$\rho_{N_f}(x, y, m; \mu) = \lim_{n\to 0} \frac{1}{\pi n} \partial_{z^*} \partial_z \log \mathcal{Z}_{N_f,n}(m, z, z^*; \mu)$$

we are naturally led to the generating functionals

$$\mathcal{Z}_{N_f,n}(m, z, z^*; \mu) = \langle \det(D + m)^{N_f} |\det(D + z)|^{2n} \rangle$$

for the eigenvalue density. The presence of conjugate quarks in the generating function induces a coupling to the chemical potential in the low-energy effective theory which is completely fixed by the pattern of chiral symmetry breaking and determines the eigenvalue density uniquely in the microscopic regime. In order to calculate the microscopic spectral density this way, one has to employ powerful integrability relations that exist for the effective partition function [18,19].

Let us compare the phases of the generating function for $n = 1$ to the regions in the complex plane characterized by a different behavior of the eigenvalue density for $N_f = 1$. For real $\hat{z}$ the phase structure follows from [20]. The extension for complex $\hat{z}$ can be obtained from the asymptotics of the microscopic generating functions given in [6]. For $|\mathrm{Re}\,(\hat{z})| > 2\hat{\mu}^2$ the generating functional is in the normal state while for $|\mathrm{Re}\,(\hat{z})| < 2\hat{\mu}^2$ two Bose condensates are separated by a first order phase transition. This phase transition occurs exactly where the oscillating region of the eigenvalue density begins, while the transition to the normal phase corresponds to the dropoff of the eigenvalue density for $|\mathrm{Re}\,(\hat{z})| > 2\hat{\mu}^2$.

Conclusions.—For QCD at nonzero chemical potential the chiral condensate is not dominated by the contribution from the smallest eigenvalues. On the contrary, we have found that the contributions from strips parallel to the imaginary eigenvalue axis do not depend on the real part of the eigenvalue as long the eigenvalue is inside the support of the spectrum. This result arises from integrating a spectral density that oscillates with a period of $1/(\Sigma V)$ and an amplitude that diverges exponentially with the volume. Although we have shown only results using the microscopic eigenvalue density, we have checked [2] that our arguments apply up to $\mu \sim 1/L$ (for $V = L^4$). In conclusion, we have uncovered a novel mechanism of chiral symmetry breaking at nonzero chemical potential where an oscillatory spectral density results in a discontinuity of the chiral condensate in the complex mass plane.

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