# Differential Growth and Instability in Elastic Shells

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Differential growth in elastic materials can produce stress either through incompatibility of growth or by interaction with the surrounding medium. In many situations, this stress can be sufficient to induce shape instability in the growing medium. To gain better insight in growth-induced instabilities, the growth of an elastic shell loaded with hydrostatic pressure or embedded in an elastic medium is studied. The residual stress arising from the incompatibility of growth and the contact stress arising from the interaction with the surrounding medium are computed with respect to growth and geometric parameters and critical values for instability are obtained. Depending on these parameters, different modes of instability can be obtained.

Nondiffusive volumetric growth, the increase of bulk mass in a given system by a source, is involved in many fundamental biological processes such as morphogenesis, physiological regulation, or pathological disorders [1]. It is, in general, a process of enormous complexity involving genetic, biochemical, and physical components at many different scales and with complex interactions. Here, we are interested in the mechanical description of volumetric growth and its ability to generate stress-related instability in soft tissues.

It has been known since the 19th century, through the work of DeVries and Sachs in developing plants, that growth can generate stresses in tissues [2]. Essentially, as growth takes place with possibly different rates at different locations within a given geometry, adjacent material points may tend to separate or overlap. The material keeps its integrity by developing *residual stresses* which remain after growth ceases. These stresses can be seen by cutting pieces of unloaded material and observing that the material changes shape as residual stress is relieved [3]. This is observed in woods and plants and is believed to play an important role in regulation of some physiological systems such as peak stress in arterial walls [4].

Stress can also develop in growing materials due to pressure induced contact. For instance, as a solid tumor grows inside a tissue, it exerts a pressure against the tissue. Accordingly, stress builds up in the tumor and is believed to inhibit growth, to lead to the collapse of the vascular system, or to modify the tumor shape [5]. This *contact stress* is not an intrinsic property as it depends both on the growth and elasticity of the tissue and the response of the surrounding medium.

An important effect of stress is its ability to generate changes in geometry and shapes through buckling-type instability. These instabilities are found in many problems in classical engineering where stress is generated by external loading [6], and they may play an important role in morphogenetic processes. For instance, it was argued in [7] that the process of primary evagination in sea urchins may be induced by a buckling instability and simplified physical models of drying gel beads seem to confirm this view [8]. Other examples where buckling-type instabilities have been associated with growth processes are the formation of pattern in fingerprints and plants, the wrinkling of leaves, organ initiation in the shoot apical meristem, shoot formation and tip growth in algae and filamentary microorganisms, the development of epithelial spheroid, the growth of blood vessels, and the formation of convolution in brain development [9]. It has been shown recently that the residual stress induced by anisotropic growth can be sufficient to create a buckling instability [10]. However, it was also found that growth or resorption can help stabilize a material by either increasing the effective thickness of the material or by creating residual stresses opposite the external load.

At the biomechanical level, soft tissues with possibly large strains and nonlinear anisotropic behavior are best represented by hyperelastic materials and modeled within the theory of finite elasticity in which their response to stress is determined by a strain energy function [1]. The modeling of such functions for tissues with given symmetries represents an important and active field of study [4,11]. Growth can be modeled by a multiplicative decomposition of the deformation gradient due to Rodriguez *et al.* [12] similar to the one found in elasto-plasticity [13]. The deformation tensor is assumed to be a product of a growth tensor describing the local evolution of a mass element with no geometric or external constraint and an elastic response of the material describing the strain necessary to ensure integrity and compatibility of the material. This theory of material growth and its various generalizations have been applied successfully to the modeling of many physiological systems such as arteries, cartilage, muscle fibers, heart tissues, and solid tumors [14]. The growth

tensor can be coupled to the strain and stress fields, the material position in the medium, the density of nutrients, or the concentration of morphogens. Of particular interest for the present study is the problem of differential growth where growth is position dependent and which is of fundamental importance in morphogenesis [15].

Here, we consider the model problem of a growing elastic shell embedded in an elastic medium. The shell and the surrounding medium are modeled by an incompressible isotropic hyperelastic material [16]. These simplifying assumptions are designed to enable us to extract important qualitative features in order to gain some insight into the processes of instabilities in a differentially growing medium.

The deformation of the material body is given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ where $\mathbf{X}$ (respectively, $\mathbf{x}$) describes the material coordinates of a point in the reference (respectively, current) configuration. Let $\mathbf{F}_{\text{inc}} = \text{Grad}(\boldsymbol{\chi})$ be the deformation gradient obtained after an incremental growth step. As described above, we assume that the gradient tensor is the product of a growth tensor $\mathbf{G}_{\text{inc}}$ by an elastic tensor $\mathbf{A}_{\text{inc}}$ so that [12]

$$\mathbf{F}_{\text{inc}} = \mathbf{A}_{\text{inc}} \cdot \mathbf{G}_{\text{inc}}. \tag{1}$$

The response of the material is given by a strain energy function $W = W(\mathbf{A})$ so that for a given elastic deformation tensor $\mathbf{A}$, the Cauchy stress tensor is $\mathbf{T} = \mathbf{A} \cdot W_{\mathbf{A}} - p\mathbf{1}$. In this relation, $W_{\mathbf{A}}$ is the derivative of $W$ with respect to (w.r.t.) $\mathbf{A}$, and $p$ is the hydrostatic pressure associated with the incompressibility constraint. The equation for mechanical equilibrium is simply given by $\text{div}(\mathbf{T}) = 0$, where the divergence is taken in the current configuration. If the body is loaded by a hydrostatic pressure $P$, the boundary condition is given by the Cauchy stress in the normal direction $\mathbf{n}$ of the boundary $\mathbf{T} \cdot \mathbf{n} = -P\mathbf{n}$.

We first consider a radially symmetric deformation of a growing shell under pressure with and without surrounding medium. The growth tensor $\mathbf{G}_{\text{inc}}$ is assumed to be isotropic but a function of the radial position $r$ in the current configuration; that is, $\mathbf{G}_{\text{inc}} = g_{\text{inc}}(r)\mathbf{1}$. The decomposition (1) represents an incremental growth step. Once (slow) growth and (fast) elastic response take place, the shell continues its growth, and after successive incremental steps the total cumulative deformation is of the form $\mathbf{F} = \mathbf{A} \cdot \mathbf{G}$, where $\mathbf{G} = g(r)\mathbf{1}$. For a given incremental growth $g_{\text{inc}}(r) = 1 + f(r - a)$ [with $f(0) = 0$, so that there is no growth at the inner boundary], a suitable form for $g(r)$ needs to be found. Since the solution for $a$ and $r$ as a function of the initial radius $R$ at each step depends on the solution of a boundary value problem (see below), there is no simple form for $g(r)$. However, a numerical study of the incremental growth process [17] reveals that for a simple linear incremental law, cumulative growth is also well modeled by a linear profile even for large variation of volumes. Therefore, in the present study, we take $g(r) = 1 + \nu(r - a)$, where $\nu$ is positive for growth and negative for resorption.

We first consider the radial deformations of the shell in the following cases: (i) a shrinking sphere in the vacuum and (ii) a growing shell inside an incompressible infinite medium. In this case, we can write in the usual coordinates $(r, \theta, \varphi)$, $\mathbf{F} = \text{diag}(\partial_R r, r/R, r/R)$, $\mathbf{A} = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, and since the material is assumed incompressible, $\alpha_2 = \alpha_3 = \alpha$ and $\alpha_1 = \alpha^{-2}$. For simplicity, we assume that both the elastic shell and the external medium are neo-Hookean materials; that is, $W_i = \mu_i(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 3)$ (where $i = \text{sh}$ or $i = \text{me}$ for shell and for medium, respectively). Let $a = r(A)$ and $b = r(B)$ be the radii in the current configuration. The incompressibility condition determines both the deformation $R^3 = A^3 + 3\int_a^r r^2/g^3(r)dr$ and the strain $\alpha(r) = r/(gR)$, up to the value of $a$ that is obtained from the boundary conditions on the Cauchy stress. The radial component of the Cauchy stress is given by

$$t_1(r) = \int_a^r \frac{\alpha}{r} \partial_\alpha [W_{\text{sh}}(\alpha^{-2}, \alpha, \alpha)]dr, \tag{2}$$

with boundary conditions $t_1(a) = 0$, and $t_1(b) = -P$ for free growth subject to a compressive hydrostatic pressure $P > 0$. When growth occurs in a medium, the Cauchy stress in the medium is given by a similar expression with $g = 1$, $W = W_{\text{me}}$ and integration from $b$ to $r$. The condition is then $t_1(\infty) = 0$ and equality of radial stresses at $r = b$. Once the radial stress is known, the deformation is completely determined and the hoop stress is given by $t_2 = t_1 + \frac{\alpha}{2}\partial_\alpha[W_{\text{sh}}(\alpha^{-2}, \alpha, \alpha)]$. Typical profiles of radial and hoop stresses for vanishing external pressure are given in Fig. 1. The constant $\nu$ is fixed by the overall increase in volume $\Delta V = (b^3 - a^3)/(B^3 - A^3)$. Resorption creates a compressive residual stress, whereas growth in an elastic medium creates both compressive contact stress (close to the outer boundary) and tensile residual stress (around the inner boundary).

The previous analysis provides a complete solution for the finite radial deformation and the question is now to study the stability of such configurations by considering axisymmetric infinitesimal deformations superimposed on the finite radial deformation. Would the residual stress created through growth and resorption be sufficient to destabilize the shell? To answer this question, we write the infinitesimal gradient tensor in the reference configuration $\mathbf{F} = (\mathbf{1} + \epsilon \mathbf{F}^{(1)}) \cdot \mathbf{F}^{(0)}$, so that $\mathbf{F}^{(1)} = \text{grad}(\chi^{(1)})$ and
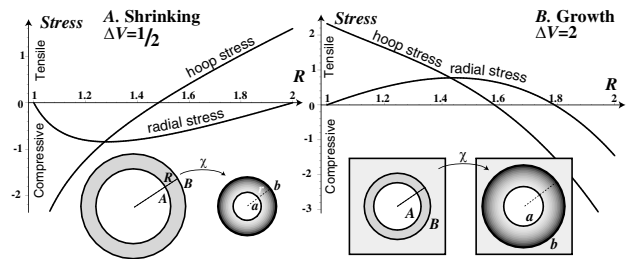


FIG. 1. Radial and hoop stresses due to (a) shrinking in the vacuum ($\Delta V = 1/2$, i.e., $\nu = -0.39$), and (b) growth inside an elastic medium ($\Delta V = 2$, i.e., $\nu = 0.378$), for an elastic shell with $A = 1$, $B = 2$ and $\mu_{\text{sh}} = \mu_{\text{me}} = 1$.

since growth is given as an outside source defined on the radial configuration, it is not affected by the perturbation and the elastic deformation tensor is $\mathbf{A} = (\mathbf{1} + \epsilon \mathbf{A}^{(1)}) \cdot \mathbf{A}^{(0)}$. Similarly, we expand the Cauchy stress $\mathbf{T} = \mathbf{T}^{(0)} + \epsilon \mathbf{T}^{(1)} + \mathrm{O}(\epsilon^2)$. The expansion of the constitutive equation leads to [10]

$$\mathbf{T}^{(0)} = \mathbf{A}^{(0)} \cdot W_{\mathbf{A}}^{(0)} - p^{(0)}\mathbf{1},$$

$$\mathbf{T}^{(1)} = \mathcal{L}{:}\mathbf{F}^{(1)} + \mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)} \cdot W_{\mathbf{A}}^{(0)} - p^{(1)}\mathbf{1},$$

$$\mathcal{L}{:}\mathbf{F}^{(1)} = \mathbf{A}^{(0)} \cdot W_{\mathbf{AA}}^{(0)}{:}\mathbf{F}^{(1)} \cdot \mathbf{A}^{(0)},$$

where $p = p^{(0)} + \epsilon p^{(1)}$, $\mathcal{L}$ is the instantaneous elastic moduli tensor (given explicitly in [16], p. 412), and $W_{\mathbf{A}}^{(0)}$, $W_{\mathbf{AA}}^{(0)}$ are the first and second derivatives of $W$ with respect to $\mathbf{A}$ evaluated on $\mathbf{A}^{(0)}$. The stability analysis proceeds by solving $\mathrm{div}(\mathbf{T}^{(1)}) = 0$.

For the growing shell, we consider axisymmetric deformation of the shell of the form $\boldsymbol{\chi}^{(1)} = [u, v, 0]^{\mathrm{T}}$, with $u$, $v$ independent of $\varphi$. The nonvanishing components of $\mathrm{div}(\mathbf{T}^{(1)}) = 0$ and the incompressibility condition form a system of 3 coupled partial differential equations of second order for $u$, $v$, and $p^{(1)}$ as a function of $(r, \theta)$ with coefficients depending on the finite-strain solution obtained to zeroth order. To solve this problem, we expand the solution in Legendre polynomials and obtain a fourth-order differential equation describing the amplitude of the field $u$ w.r.t. the $n$th Legendre polynomial. This equation generalizes the classical one for the stability of nongrowing shells under pressure [16]. For each mode $n$, the solution of this boundary value problem is possible only for a particular combination of parameters $a = a(\nu, \delta, n)$, where $\delta = (b - a)$ is the width of the shell in the current configuration. These values of $a$ are the critical values of the inner radius where the $n$th mode first appears. For small $\delta$, these curves in parameter space can be obtained by perturbation expansion in the form $a = a_0 + a_1\delta + a_2\delta^2 + \mathrm{O}(\delta^3)$, where $a_0$ is the first positive solution of $(n + 2)(n - 1)a_0^{12} + 2(n^2 + n + 7)a_0^6 = 3n(1 - n)$, $a_1 = 1/2a_0^3 + 1/2\nu a_0 - 1/2$, and $a_2$ is given by a similar expression in terms of $\nu$ and $a_0$ (but too long to be given here). The method is given in [10], and its validity and domain are checked numerically by the determinant method for fourth-order linear boundary value problems [18,19]. Once the bifurcation of each individual mode is known, the critical value $a_c$ where the shell first bifurcates (independently of the mode number) is

$$a_c = 1 - \left(\frac{1}{3} - \frac{1}{2}\nu\right)\delta - \frac{1}{24}(2 + 4\nu + 5\nu^2)\delta^2 + \mathrm{O}(\delta^3).$$

For a given $\delta$, the value of $a_c$ and $b_c = a_c + \delta$ can be used in Eq. (2) to solve $t_1(b_c) = -P$ with respect to $\nu$. The value of $\nu_c = \nu(a_c, b_c)$ can then be used to compute the initial radii $A$ and $B$, hence, the current and initial thicknesses (as shown in Fig. 2). The selected mode shown in Fig. 2 is given by the largest integer $n$ less than

$$N = \sqrt{\frac{3}{\delta}}\left(1 - \sqrt{\frac{\delta}{12} + \frac{8 + 3\nu}{24}\delta - \frac{29\nu^2 + 220\nu + 282}{384}\delta^2}\right).$$

The mode selected at the bifurcation depends on the initial thickness in such a way that thinner shells become unstable with an increasingly high mode number as found in classical shells under compression (albeit for different thickness values). The basic physical process of the instability shown in Fig. 2 is as follows: A shell with no external loading of thickness $A/B$ becomes unstable when both its current thickness $a/b$ is small enough and sufficient compressive residual stress has built up in the shell. Under a pressure $P_n$ chosen to be the critical pressure for mode $n$, the shrinking shell becomes unstable with possible mode $m$, $2 \leq m \leq n$.

We now turn our attention to the case of a growing shell inside an incompressible elastic medium. The stability analysis in such interfacial problems is delicate due to the matching of stresses at the interface in the deformed configuration. Nevertheless, some insight can be gained by modeling the problem by assuming that the outside medium creates a hydrostatic compressive force equivalent to the one created through growth and contact. In this context, we compute a lower bound for instability in terms of the initial thickness and the volume increase for a different relative medium response $\Delta\mu = \frac{\mu_{\mathrm{me}}}{\mu_{\mathrm{sh}}}$. The linearized equation for the instability threshold is the same one as in the previous situation, and the same estimate for small current thicknesses can be used. For each $\delta$, the value of $\nu$ necessary for the radial problem to satisfy the boundary values can be computed. Once such a value is known, the initial thickness and volume of the current configuration can be found for different values of $\Delta\mu$ as shown in Fig. 3.
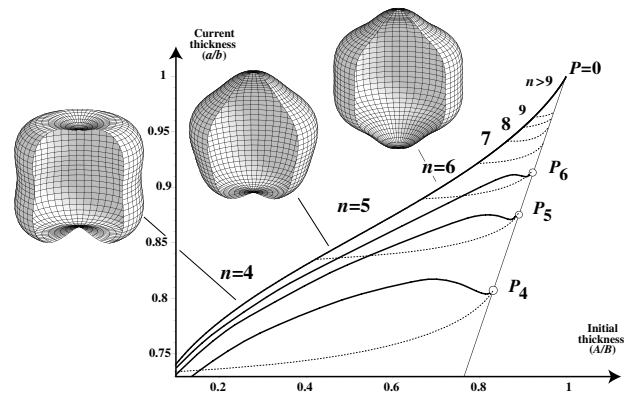


FIG. 2. Instability of the shrinking shell. For an initial thickness $A/B$, the shell first becomes unstable due to residual stress when the thickness $a/b$ reaches the critical value given by the solid curves. The solid curves are obtained at constant pressure $P_i$ given by the critical value for the instability of the $i$th mode without growth. The dashed lines denote the regions of the mode of deformation $n$ found at the bifurcation (the corresponding configurations $n = 4, 5, 6$ are shown, but their amplitude is not predicted by the theory).
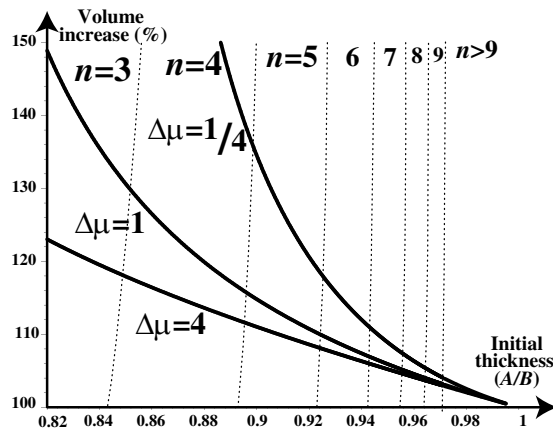
FIG. 3. Instability of a growing embedded shell. For an initial thickness $A/B$, the shell first becomes unstable due to contact stress when the volume increases ($100v/V$) reaching the critical value given by the solid curves (with different relative medium responses $\Delta\mu = 1/4, 1, 4$). The dashed lines represent the different instability modes.

The analysis presented in this Letter shows the effect of differential growth on the stability of shells. In a shrinking shell, two effects drive the instability, the shell thickness decreases and the residual stress is compressive. Not surprisingly, in the absence of hydrostatic pressure, the neo-Hookean shell is stable under moderate shrinking and becomes unstable only when a large portion of the initial volume has been resorbed. Under compressive loads, the shell can rapidly become unstable with a lower or equal mode number than the one given by the initial thickness. In the case of a growing shell creating compressive stress through contact, the instability for a thin shell is driven rapidly by the volume increases and the rigidity of the external medium.

The assumptions used in this Letter were chosen for the purpose of illustrating the general problem of stability in a growing medium and identifying the basic mechanical processes. Clearly, these assumptions do not correspond to realistic tissues and many more sophisticated models can be used for specific physiological systems such as more refined constitutive laws for both tissues and the external medium. Nevertheless, we have shown that resorption or growth can induce sufficient stress to render the growing tissue unstable even in the absence of external loading, a process that could be of fundamental importance in morphogenesis. Similarly, growth can induce stresses which help stabilize tissues as they operate in physiological regimes.

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