Breakdown of Kolmogorov Scaling in Models of Cluster Aggregation

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(Received 5 October 2004; published 17 May 2005)

We describe a model of cluster aggregation with a source which provides a rare example of an analytically tractable turbulent system. The steady state is characterized by a constant mass flux from small masses to large. Thus it can be studied using a phenomenological theory, inspired by Kolmogorov's 1941 theory, which assumes constant flux and self-similarity. We prove that such self-similarity is violated in dimensions less than or equal to two. We then use dynamical renormalization group techniques to show that the scaling of multipoint correlation functions implies nontrivial multifractality. The analytical results are supported by Monte Carlo simulations.

DOI: 10.1103/PhysRevLett.94.194503 PACS numbers: 47.27.Gs, 05.10.Cc, 05.70.Ln, 68.43.Jk

A major challenge in theoretical physics is to develop a formalism to calculate the statistical properties of many body systems that are far from equilibrium. An important subset of problems that have attracted much interest are those of the turbulent type, the defining characteristic being the existence of a stationary state with widely separated sources and sinks of some conserved quantity. Stationary states of turbulent systems lack detailed balance. Instead they are characterized by a flux, mediated by nonlinear interactions, of the conserved quantity from source to sink. Universal statistics are expected in the region far from sources and sinks, also known as the inertial range. The best known example is the solution of the Navier-Stokes (N-S) equations at a high Reynolds number with large scaling force [1]. It is characterized by an energy flux from large scales to small where viscosity dissipates the energy. Other examples include Burgers turbulence [2], the Kraichnan model of passive scalar advection [2], wave turbulence [3], and kinematic magneto hydrodynamics [2]. All such systems admit a phenomenological description based on constant flux and the assumption of self-similarity. The first such theory was the Kolmogorov 1941 self-similarity (SS) theory of N-S turbulence. Understanding the limitations of such phenomenology has preoccupied theorists ever since. Analytic progress has been slow despite strong numerical and experimental evidence for a nontrivial breakdown of self-similarity. This is because the N-S equations lack an obvious small parameter which would permit a perturbative treatment. Studies of other systems, notably Burgers equation and the Kraichnan model, have been more successful [2], providing insight into the breakdown of self-similarity. In this Letter we add to this small class of analytically tractable turbulence models by characterizing the breakdown of self-similarity in a system of diffusing coagulating masses in the presence of a source.

We begin by defining our model and explaining why it is turbulent in the above sense. Consider a d-dimensional cubic lattice with sites occupied by particles of positive mass. From a given configuration, the system evolves as follows. At rate D, a particle hops to one of its $2d$ nearest neighbor sites. At rate 2λ, two particles on the same site with masses m_1 and m_2 coalesce into a new particle with mass $m_1 + m_2$. At rate J/m_0, particles of mass m_0 are injected at each site. The initial configuration has no masses. J is the average mass flux into the system. D and λ are assumed to be independent of mass. We choose to work with $\lambda < \infty$ as it is more amenable than $\lambda = \infty$ (infinite reaction rate) to a field theoretic treatment. However, the two cases belong to the same universality class. We call this model the mass model (MM). A generalization is a model where the masses (or "charges") can be positive or negative. Here, with rate J_c/m_0^2 particles of charge m_0 and $-m_0$ are injected. J_c is average influx of squared charge. We call this model the charge model (CM). We are interested in the continuous limit of these models. The steady state has a constant mass flux from small masses to large via coagulation. The inertial range is $m \gg m_0$. The mass flux is analogous to the energy flux in N-S turbulence. The mass density, N_m, is analogous to the spectral energy density, $E(k)$. Let $C_n(m_1, \ldots, m_n)(\Delta V)^n \prod_i dm_i$ be the probability of having particles of masses in the intervals $[m_i, m_i + dm_i]$ in a volume ΔV. How does $C_n(m_1, \ldots, m_n)$ vary with mass when $m_1, \ldots, m_n \gg m_0$? In particular what is the homogeneity exponent γ_n defined from $C_n(\Gamma m_1, \ldots, \Gamma m_n) = \Gamma^{-\gamma_n} C_n(m_1, \ldots, m_n)$? The loose analogue of $C_n(m_1, \ldots, m_n)$ in N-S turbulence is $\langle E_{k_1} \ldots E_{k_n} \rangle$. This analogy is closer if one models N-S turbulence using shell models. Also, the mean field limit of MM is structurally similar to the kinetic equation of three-wave weak turbulence [3].

Having discussed the MM in a turbulence context we should point out that it is also of independent interest in other branches of physics. Examples include submonolayer epitaxial thin film growth [4], river networks [5,6], force fluctuations in granular bead packs [7], and nonequilibrium phase transitions [8,9]. It also maps [10] onto the directed Abelian sandpile model of self-organized criticality, one of the earliest models to generate power laws from simple dynamical rules [11]. Because of its wide applicability, much is already known about the physics of the MM. It was shown in Refs. [12–14] that in one dimension $\gamma_1 = 4/3$ for MM and $\gamma_1 = 5/3$ for CM. By studying the two point correlations, it was shown that in $d < 2$, $\gamma_1 = (2d+2)/(d+2)$ for MM and $\gamma_1 = (3d+2)/(d+2)$ for CM [15,16].

Less is known about the γ_n for $n > 1$. We address this in the present work. We find that the scaling of the C_n in the MM points to a nontrivial multifractal structure in $d < 2$. From the turbulence perspective, this means that the SS theory of the MM breaks down at large masses in $d < 2$. In this Letter we first derive the SS predictions for γ_n based on an assumption of self-similarity. We then confirm the multifractal structure of the MM by an exact computation of the exponent, γ_2. We then present a dynamical renormalization group computation of γ_n in $d < 2$ for any n as an expansion in powers of $\epsilon = 2 - d$ and thus obtain the corrections to the SS exponents. In $d = 2$ the SS predictions acquire logarithmic corrections.

We first calculate γ_n from a SS theory. Assume that C_n depends only on the m_i, J, and D. The dimensions of these parameters are $[J] = ML^{-d}T^{-1}$, $[D] = L^2T^{-1}$, $[C_n] = L^{-nd}M^{-n}$, and $[m] = M$. Dimensional analysis gives $C_n \sim (JD^{-1})^{nd/(d+2)} m^{-\gamma_n^{\mathrm{SS}}}$, where

$$\gamma_n^{\mathrm{SS}} = \left(\frac{2d+2}{d+2}\right)n \tag{1}$$

is the SS exponent. γ_n^{SS} is linear in n, reflecting the assumed self-similarity of the statistics. When $n = 1$, $\gamma_1^{\mathrm{SS}} = (2d+2)/(d+2)$ coincides with the result of an exact computation of γ_1 for $d < 2$ [16]. The self-similarity conjecture assumes that C_n does not depend on λ, m_0, the lattice spacing, and the box size $\Delta V dm_1 \ldots dm_n$. It is independent of lattice spacing because the effective field theory describing MM in $d < 2$ is renormalizable. We will, however, find an anomalous dependence of correlation functions on a length scale depending on the other parameters leading to a violation of self-similarity.

Before computing multipoint correlations we first outline the technical machinery such computations require. Starting from the lattice model, an effective field theory of MM may be constructed using the formalism of Doi and Zeldovich [17,18]. It is then possible to establish an exact map from this field theory to the following stochastic integro-differential equation [19,20]:

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)\phi(m) = \lambda\int_0^m dm'\phi(m')\phi(m-m') - 2\lambda\phi(m) \times \int_0^\infty dm'\phi(m') + \frac{J}{m_0}\delta(m-m_0) + i\sqrt{2\lambda}\phi(m)\eta, \tag{2}$$

where $\eta(\vec{x},t)$ is white noise with $\langle\eta(\vec{x},t)\eta(\vec{x}',t')\rangle = \delta(t-t')\delta^d(\vec{x}-\vec{x}')$ and $i = \sqrt{-1}$. The imaginary noise accounts for interparticle correlations [19,20]. All correlation functions of the mass distribution can be expressed in terms of the correlation functions of $\phi(m,\vec{x},t)$. In particular,

$$C_n(m) \sim \frac{1}{n!}\langle[\phi(m,\vec{x},t)]^n\rangle\left[1 + O\left(\frac{1}{m}\right)\right], \tag{3}$$

where $\langle\ldots\rangle$ denotes averaging with respect to noise η [21].

Equation (2) simplifies after taking Laplace transform with respect to the mass variable [20]. Let $R_\mu(\vec{x},t) = \int_0^\infty dm\phi(m,\vec{x},t) - \int_0^\infty dm\phi(\vec{x},m,t)e^{-\mu m}$. Then,

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right)R_\mu = -\lambda R_\mu^2 + \frac{j}{m_0} + i\sqrt{2\lambda}R_\mu\eta(\vec{x},t), \tag{4}$$

where $j = J(1 - e^{-\mu m_0})$. Equation (4) is the stochastic rate equation for the one particle model $A + A \to A$ with a source [22]. To compute $C_n(m,t)$, correlation functions of the form $\langle R_{\mu_1}(\vec{x},t)R_{\mu_2}(\vec{x},t)\cdots R_{\mu_n}(\vec{x},t)\rangle$ are needed. These are nontrivial, as the $R_\mu(\vec{x},t)$'s are correlated for different μ's via the common noise term in Eq. (4).

As promised, one can confirm that $\gamma_n \neq \gamma_n^{\mathrm{SS}}$ in $d < 2$ by computing γ_2 exactly. From the definition of $\gamma(2)$, it follows that $\langle R_{\mu_1}R_{\mu_2}\rangle = (\mu_1\mu_2)^{\gamma_2/2-1}\psi(\mu_1/\mu_2)$, where $\psi(x)$ is an unknown function satisfying $\psi(x) = \psi(1/x)$. We need the $\mu_1, \mu_2 \to 0$ behavior of $\langle R_{\mu_1}R_{\mu_2}\rangle$. Averaging Eq. (4) with respect to η and setting $\partial_t\langle R_\mu\rangle = 0$ in the large time limit, we find $\langle R_\mu R_\mu\rangle = j/(\lambda m_0) \approx J\mu/\lambda$ for $\mu \ll m_0^{-1}$. Comparing with the above scaling form, we get $\gamma_2 = 3$ exactly, independent of the dimension d. This result establishes the multifractality of the MM: γ_0, γ_1, and γ_2 are not on a straight line. $\gamma_2 = 3$ is the analogue of the 4/5 law of N-S turbulence [1]. In both cases the balance of flux and nonlinearity in the stationary state fixes the scaling of the particular correlator responsible for flux transfer. An important difference is that the 4/5 law for N-S respects SS. For the MM it does not. We verified these results with Monte Carlo simulations of the MM in one dimension, measuring the γ_n for $n = 1$, 2, 3, and 4. The results are shown in Fig. 1. The numerical exponents agree with the exact values of γ_1 and γ_2 and for larger n, show nontrivial multifractality. Consider now how to access higher order γ_n when we have no constant flux relation. We first show that we cannot compute them with a small noise expansion of Eq. (2).

The Feynman rules for perturbative expansion of correlation functions [23] are summarized in Fig. 2. They follow from iteration of Eq. (2) with respect to λ and averaging over noise. The n-point correlation function $\langle\prod_{i=1}^n R_{\mu_i}(\vec{x}_i,t_i)\rangle$ is the sum of all diagrams with n outgoing lines built from the blocks of Fig. 2. It is convenient to first sum all tree diagrams. Let R_{mf}, denoted by a thick line with a cross, be the sum of all tree diagrams with one outgoing line. R_{mf} satisfies the diagrammatic equation in Fig. 3(a), corresponding to the noiseless limit of Eq. (4). The solution is

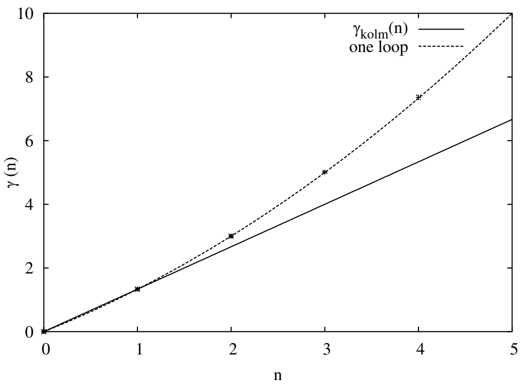

FIG. 1. γ_n as a function of n in one dimension. The dotted line shows Eq. (9) with $\epsilon = 1$ and terms of order ϵ^2 and higher set to zero. The values of γ_0, γ_1, and γ_2 are exact. $\gamma(3)$ and $\gamma(4)$ were obtained by Monte Carlo simulations performed on a lattice of size 10^5 and averaged 2×10^7 times with $J = 4D$.

$$R_{mf}(t) = \sqrt{\frac{j}{m_0 \lambda}} \tanh\left(\sqrt{\frac{j\lambda}{m_0}} t\right) \overset{t\to\infty}{\longrightarrow} \sqrt{\frac{j}{m_0\lambda}}. \qquad (5)$$

Figure 3(b) shows the equation for the tree-level propagator, $G_{mf}(x_2 t_2; x_1 t_1)$. The solution is

$$\frac{G_{mf}(\mathbf{2};\mathbf{1})}{G_0(\mathbf{2};\mathbf{1})} = \left[\frac{\cosh\sqrt{\frac{j\lambda}{m_0}} t_1}{\cosh\sqrt{\frac{j\lambda}{m_0}} t_2}\right]^2 \overset{t_{1,2}\to\infty}{\longrightarrow} e^{-\Omega(t_2 - t_1)}, \qquad (6)$$

where G_0 is the diffusive Green's function, $\mathbf{2} = (\vec{x}_2, t_2)$, $\mathbf{1} = (\vec{x}_1, t_1)$, and $\Omega = 2\sqrt{j\lambda/m_0}$ is the inverse mean field response time. Loop diagrams constructed from the vertices of Fig. 2, G_{mf} and R_{mf} are finite in $d < 2$.

The small noise expansion of C_n is the loop expansion around R_{mf}^n. We now show that the loop expansion is a weak coupling expansion in λ. Consider a contribution to $\langle R_\mu^n \rangle$ with L loops, V vertices and N R_{mf} lines. The λ factors arise from Ld momentum integrals ($\lambda^{Ld/4}$), V time integrals ($\lambda^{-V/2}$), N R_{mf} lines ($\lambda^{-N/2}$), and V vertices (λ^V). $V = L + N - n$ since there are $N - n$ triangular vertices and L quartic vertices. Thus, an L-loop contribution to $\langle R_\mu^n \rangle$ is proportional to $\lambda^{-n/2 + L(d+2)/4}$. Therefore the loop expansion corresponds to the perturbative expansion of $\langle R_\mu^n \rangle$ around R_{mf}^n with parameter $\lambda^{(2+d)/4}$. The conditions whereby the loop corrections can be neglected are derived using dimensional analysis. The scale of diffusive fluctuations is given by the only constant of dimension length obtainable from D and j/m_0: $L_D = (m_0 D/j)^{1/(d+2)}$. The dimensionless parameter in the loop expansion is then $g(L_D) = \lambda L_D^\epsilon$, where $\epsilon = 2 - d$. The $m \to \infty$ behavior of $C_n(m)$ is given by the $\mu \to 0$ behavior of $\langle R_\mu^n \rangle$. In $d < 2$, $g_0 \to \infty$ when $\mu \to 0$ and the expansion breaks down. Computation of the γ_n therefore requires resummation of infinitely many terms. This was done using dynamical renormalization group (RG) formalism.

We first examine $\langle R_\mu \rangle$. References [19,22] showed that the resummation of the leading terms in the loop expansion renormalizes the coupling constant. The renormalization law can be found exactly, $\lambda \to \lambda_R = C(\epsilon) L_D (1 + O(\mu))$, where $C(\epsilon)$ is a dimensionless constant. Replacing λ by λ_R in Eq. (5) gives $\langle R_\mu \rangle$. Therefore $R_\mu \sim (J\mu)^{d/(d+2)}$ as $\mu \to 0$. The inverse Laplace transform gives

$$C_1(m) \sim (D^{-1}J)^{d/(d+2)} m^{-\gamma_1}, \qquad d < 2, \qquad (7)$$

where $\gamma_1 = (2d + 2)/(d + 2)$. This agrees with the SS prediction and the exact result. The SS conjecture thus corresponds to renormalized mean field theory. A dimensional argument gives the correct scaling of R_μ since it has zero anomalous dimension. Hence $\langle R_\mu \rangle$ scales with its physical dimension. This is untrue for higher order correlation functions.

Consider the diagrams contributing to $\langle R_{\mu_1} R_{\mu_2} \rangle$ up to one-loop order shown in Fig. 4. Figures 3 and 4 contribute to coupling constant renormalization of the mean field contribution of Fig. 1. Figure 2 is of a different nature: it generates the order ϵ contribution to the anomalous dimension of $\langle R_{\mu_1} R_{\mu_2} \rangle$. There are $\binom{n}{2}$ such diagrams contributing to the anomalous dimension of the n-point function. Calculating these diagrams gives the anomalous dimension of the n-point function as $-\epsilon n(n-1)/2$. The physical dimension is $-dn$. The scaling dimension is the sum of the two. Hence $\langle \prod_{i=1}^n R_{\mu_i} \rangle \sim \Phi_n \prod_{i=1}^n L_i^{-d-\epsilon(n-1)/2}$, where Φ_n is a scaling function of the variables L_i/L_j, where $L_i = [m_0 D/j(\mu_i)]^{1/(d+2)}$. Inverse Laplace transform gives

$$C_n(m_1, \ldots, m_n) \sim \Psi_n \prod_{i=1}^n \frac{1}{m_i} \left(\frac{J}{D m_i}\right)^{(d+\epsilon(n-1)/2)/d+2}; \quad (8)$$

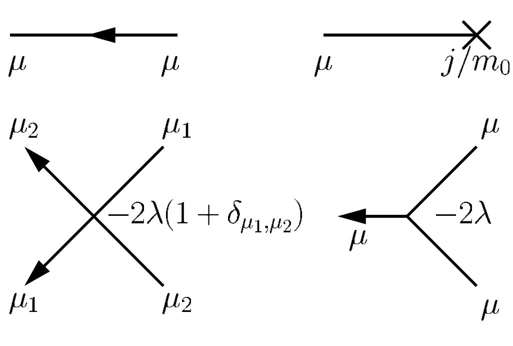

FIG. 2. Propagators and vertices of the theory.

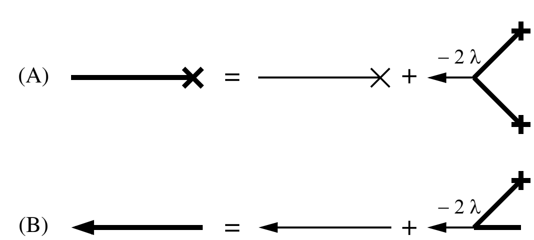

FIG. 3. Diagrammatic equations for: (a) R_{mf} and (a) G_{mf}.

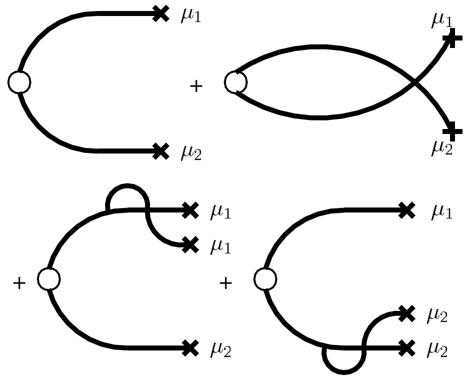

FIG. 4. Zero and one-loop contributions to $\langle R_{\mu_1} R_{\mu_2} \rangle$. Circles signify lines terminating at the same spatial point.

Ψ_n is a scaling function of the variables m_i/m_j and the parameters ΔV, m_0 and λ. In the limit when $m_0 \to 0$, and $\lambda \to \infty$, we expect $\Psi_n \sim (\Delta V)^{\epsilon n(n-1)/2d}$. Note that $\Psi_n \to 0$ as $\Delta V \to 0$ which describes anticorrelation of particles. From Eq. (8),

$$\gamma_n = \left(\frac{2d+2}{d+2}\right)n + \left(\frac{\epsilon}{d+2}\right)\frac{n(n-1)}{2} + O(\epsilon^2). \qquad (9)$$

The first term is γ_n^{SS} and the nonlinear terms correct SS scaling leading to a breakdown of self-similarity. The predictions up to $o(\epsilon)$ are also shown on Fig. 1. The closeness is surprising for a first order ϵ-expansion with $\epsilon = 1$. Note that the first order ϵ-expansion matches the exact answers for γ_1 and γ_2. We conjecture our one-loop answer for γ_n is exact in one dimension and hope to prove this using the methods of [24]. The quadratic nature of the corrections in Eq. (9) is different from the analogous quadratic scaling exponents of velocity structure functions in the lognormal model of random energy cascade of N-S turbulence. For large n, the lognormal model predicts negative scaling exponents which violate the Novikov inequality [1]. In contrast Eq. (9) predicts that $\gamma(n)$ is greater than the SS value, reflecting the anticorrelation between particles.

In $d = 2$, nonlinear logarithmic corrections to SS are expected. These can be calculated exactly using the RG method. They vanish for $n = 2$ consistent with the exact result $C_2(m) \sim m^{-3}$. We present the final results here:

$$C_n(m) \sim \frac{J^{n/2}[\ln(m)]^{n-n^2/2}}{m^{3n/2}}\left[1 + O\left(\frac{1}{\ln(m)}\right)\right], \qquad d = 2. \qquad (10)$$

It is straightforward to extend the above results to CM. We skip the details and present the final result:

$$^{\mathrm{CM}}\gamma_n = {}^{\mathrm{CM}}\gamma_n^{\mathrm{SS}} + \left(\frac{\epsilon}{d+2}\right)n(n-1) + O(\epsilon^2), \qquad (11)$$

where ${}^{\mathrm{CM}}\gamma_n^{\mathrm{SS}} = (3d+2)/(d+2)n$ is the SS prediction. As in MM, the $O(\epsilon^2)$ terms vanish for $n = 1$ and 2, and SS theory fails to predict the scaling of multi point correlation functions. ${}^{\mathrm{CM}}\gamma_2 = 4$ is exact in all dimensions reflecting the conservation of flux of squared charge.

As in N-S turbulence, dimensional analysis is too rough to capture the detail of interactions in MM. It misses the recurrence of random walks in $d < 2$. This creates anticorrelation between particles: nearby particles meet infinitely often and are thus very likely to merge, leaving only one particle. This effective repulsion between particles causes the probability of finding n particles in volume ΔV to go to zero faster than ΔV. Multiparticle configurations are therefore much less probable than in a self-similar theory.

We thank A. Kupiainen and G. Falkovich for useful discussions. We acknowledge support from NSF Grant No. DMR-0207106 (R. R.) and Marie Curie Grant No. HPMF-CT-2002-02004 (C. C.). O. Z. thanks the hospitality at Brandeis University, where part of this work was done.

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