# Kinematic Characterization of Valvular Opening

Gianni Pedrizzetti*

*Dipartimento Ingegneria Civile, Università di Trieste. P.le Europa 1, 34127 Trieste, Italy*

(Received 16 December 2004; published 17 May 2005)

The evaluation of the valvular opening due to a pulse flow, as is the case of cardiac valves, requires the knowledge of the leaflets material properties and the coupled solution of the fluid and solid equations. This approach is not commonly feasible. A different approach is introduced here to describe the opening behavior of valvular leaflets by a functional kinematic relationship. The asymptotic analysis, in the limit of leaflet opening without vortex shedding, is presented for a two-dimensional rigid leaflet model under the irrotational scheme. The approach is then verified by numerical solution of the Navier-Stokes equation in asymptotic and nonasymptotic conditions.

DOI: 10.1103/PhysRevLett.94.194502 PACS numbers: 47.60.+i, 47.11.+j, 47.32.Ff, 87.19.Uv

Phenomena involving the dynamic interaction between a fluid and a movable boundary are encountered in many applied contexts. Such interactions are common in the cardiovascular network, where they range from the flow over a compliant elastic wall to the flow in the presence of a moving valve. This work was originally motivated by the need to understand and model the pulsed flow through the bileaflet mitral valve, that the blood crosses when passing from the left atrium to the left ventricle of a human heart. Similar problems arise in industrial devices involving the flow passing through orifices with movable doors that prevent a return flow.

Commonly, when the flow pushes onto the upstream face of a closed valve the leaflets open rapidly with the fluid making little resistance to it. The mathematical modeling of a fluid-valve interaction is made complicated by the large motion of the thin leaflets within the flow domain and their rapid response to the force exerted by the accelerating fluid. A rigorous mathematical model should be based on the equations of fluid mechanics for the flow, those of solid mechanics for the structure, the proper coupling relations, and tackle the problem of the solution of the resulting system. Recent numerical studies [1,2] began to consider such complete numerical approaches, that employ finite elements and a coupling strategy, to reproduce the flow in models of the aortic valve. A numerical technique dealing with a simplified kind of interaction was introduced previously (see [3]). That approach is interesting and, in principle, mathematically well posed; however, when it is applied to discrete systems, it gives rise to unrealistic distributions of stress on the solid.

An essential ingredient of a complete numerical modeling is the proper mechanical description of the valvular large deformation regime and the knowledge of the actual nonlinear elasticity parameters, commonly neither homogeneous nor isotropic. A homogeneous and isotropic Neo-Hookean elastic structure was used in [1] as a possible first approximation for the aortic valve. A linearly elastic leaflet, with no resistance at the constraint, is considered in the model problem of [2]. In general, either the valve material behavior and its parameters are unknown and quite difficult to assess in physiological applications. For this reason, in addition to the modeling difficulties, the study of fluid flows in presence of moving leaflets often appears as a prohibitive task.

An asymptotic representation of the leaflets dynamics is introduced here. It follows from a fluid dynamics analysis of the fluid-solid interaction and does not depend on either the material structure or its elastic properties. After accounting that a leaflet is a structure where vortex shedding, if any, occurs from the trailing edge only, we consider here the limiting condition when the leaflet moves such that there is no vortex shedding from the leaflet trailing edge. In fact, vortex shedding would produce a pressure drop at the edge with the result of a reduction of pressure on the downstream face and an increase of the force acting on the leaflet [4] to possibly accelerate it. When the leaflet moves fast enough, vortex shedding is inhibited, while the leaflet cannot move faster than this, e.g., pushing the flow instead of making resistance, unless externally activated. In other words, the no shedding assumption corresponds to considering the fastest possible motion of a massless leaflet.

The asymptotic assumption is in qualitative agreement with the results from the cited numerical simulations [1,2] as well as experimental observations [5–8] that showed how, during the opening phase, a normal physiologic leaflet moves approximately with the fluid without significant generation of vorticity downstream until it approaches its end of stroke position. Therefore the no shedding assumption represents limiting conditions that well approximate the normal valvular opening.

The analysis is applied here to the case of a rectilinear leaflet in order to make the mathematics as simple as possible, still retaining the complete modeling issues. Therefore the results do not explicitly apply to the mitral valve. The present analysis is theoretical and finalized to the description of a method that can be applied in the study of many specific applications.

With this in mind, we consider a simple model problem in a two-dimensional half channel of width $H$ and a pulsed flow of period $T$. Taking $H$ and $T$ as the unit of length and time, respectively, the dimensionless flowing discharge is assumed to be $V(t) = \mathrm{Sr}^{-1}[1 - \cos(2\pi t)]/2$, where $\mathrm{Sr} = H/(V_{\mathrm{peak}}T)$ is the Strouhal number and $V_{\mathrm{peak}}$ is the dimensional section-averaged peak velocity. A channel Reynolds number is $\mathrm{Re} = HV_{\mathrm{peak}}/\nu$, where $\nu$ is the kinematic viscosity of the fluid. A rectilinear leaflet of unitary length can rotate clockwise about one end kept hinged on the inferior wall; the angle between the plate and the wall is indicated with $\theta(t)$ such that when $\theta = \pi/2$ the channel is closed by the plate and when $\theta = 0$ the plate lays on the wall. The geometry is sketched in Fig. 1. The points along the plate are identified by a material coordinate $s$ ranging from $s = 0$ at the wall hinge to $s = 1$ at the trailing edge. The complex number $z = x + iy$ is used to indicate a point in the physical plane; the leaflet is thus found on $z = se^{i\theta}$. When the flow starts, the channel is closed with the plate in the vertical position, $\theta(0) = \pi/2$.

The problem can be solved in the limit of high Reynolds number, $\mathrm{Re} \to \infty$. In this limit the flow is irrotational with the exception of possible rotational singularities [4,9]. The undisturbed velocity profile is uniform, $v_x(t, x, y) = V(t)$, with a zero-thickness boundary layer on the inferior wall. The additional velocity due to the presence of the rotating plate can be represented as that of a vortex sheet whose strength $\gamma(t, s)$ is chosen to satisfy the impermeability boundary condition on the plate itself. This condition is transformed, following the panel method approach [10], in the integral equation, where the time dependence is omitted for brevity,

$$\int_0^1 \gamma(s')\mathbb{K}(s, s')ds' + s\dot{\theta} = -V\sin\theta. \qquad (1)$$

Equation (1) expresses, for each $s$, that the instantaneous fluid velocity normal to the plate, given by the sum of the undisturbed velocity, $V(t)\sin\theta$, plus the contribution by the vortex sheet, expressed by the integral on the left-hand side, must be equal to the velocity due to the leaflet rotation $\dot{\theta} = d\theta/dt$. The integrand $\gamma(s')\mathbb{K}(s, s')ds'$ represents the velocity normal to the leaflet in $s$ due to an infinitesimal segment of vortex sheet centered in $s'$. In (1) the kernel,

$$\mathbb{K}(s, s') = -\frac{1}{4}\Im\left\{e^{i\theta}\cot\left[\frac{\pi i}{2}e^{i\theta}(s' - s)\right] - e^{i\theta}\cot\left[\frac{\pi i}{2}(s'e^{-i\theta} - se^{i\theta})\right]\right\}, \qquad (2)$$

represents the velocity induced in $s$ by a unitary vortex in

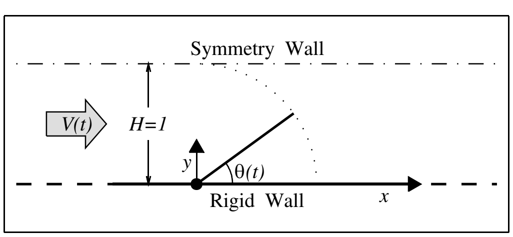


FIG. 1. Sketch of the model problem.

$s'$, its symmetric image with respect to $y = 0$ to guarantee a zero normal velocity on the lower wall, and their infinitely periodic—period 2—images along $y$ to account for the zero normal velocity in $y = 1$ [11]. The operator $\Im$, the imaginary part of its argument, is employed to extract the velocity component normal to the leaflet after the complex velocity is rotated by $e^{i\theta}$.

The resulting irrotational velocity field presents a square root singularity at the trailing edge [4,9]; this singularity does not exist in a real flow with arbitrarily small but nonzero viscosity. Mathematically, the singularity is removed by imposing the Kutta condition that the velocity is finite at the edge and is written here as $\gamma(t, 1) = 0$. This condition is commonly satisfied, physically, by a shedding of vorticity from the trailing edge and the consequent establishment of a pressure drop between the two faces of the edge [4,9]. Alternatively, as considered in the present asymptotic approach, the leaflet can rotate such that there is no shedding, the motion itself, the value of $\dot{\theta}$, is such to satisfy the Kutta condition.

The solution to (1) is found numerically by a panel method approach [10]. The vortex sheet is divided into $N$ segments, of length $1/N$, centered in $s_k = (k - 1/2)/N$, of constant strength $\gamma_k$, $k = 1 \ldots N$. The integral of the kernel (2) is evaluated analytically on each constant-strength segment. The discrete version of Eq. (1) is satisfied on each point $s_k$ and becomes a set of $N$ linear equations on the $N + 1$ unknowns $\gamma_k$ and $\dot{\theta}$. The Kutta condition, written in terms of the last three $\gamma_k$ by quadratic extrapolation, becomes the $N + 1^{\mathrm{th}}$ linear equation. The solution of the linear system does not change appreciably when $N$ is varied from 128 to 1024. The resulting distribution $\gamma(s)$ is zero at $s = 0$, 1 and presents one maximum whose position moves towards smaller values of $s$ when $\theta \to 0$. Equation (1) also shows that the solution depends linearly on the instantaneous value of $V(t)$. The normalized angular velocity $\dot{\theta}/V$ is thus a function of $\theta$ only; this dependence is a characteristic curve of the leaflet $\dot{\theta}/V = f(\theta)$. The solution for this geometry is reported in Fig. 2.

When the plate rotates in agreement with its characteristic curve, the Kutta condition is instantaneously satisfied by the proper angular velocity, $\dot{\theta}$, and the pressure is continuous at the trailing edge. When the leaflet rotates slower, not fast enough to satisfy the Kutta condition, a vortex shedding develops and creates an additional opening force (by lowering pressure on the downstream face). This would occur when the body finds resistance (for its mass, elasticity, viscosity) and is unable to follow the asymptotic regime. On the opposite, if the leaflet rotates more rapidly (for an external activation, or for its inertia during flow deceleration) an upstream vortex shedding would induce a positive pressure jump to decelerate its motion.

The corresponding leaflet dynamics, $\theta(t)$, corresponding to the time law $V(t)$, depends on the value of the Strouhal number. The time evolution is computed by numerical

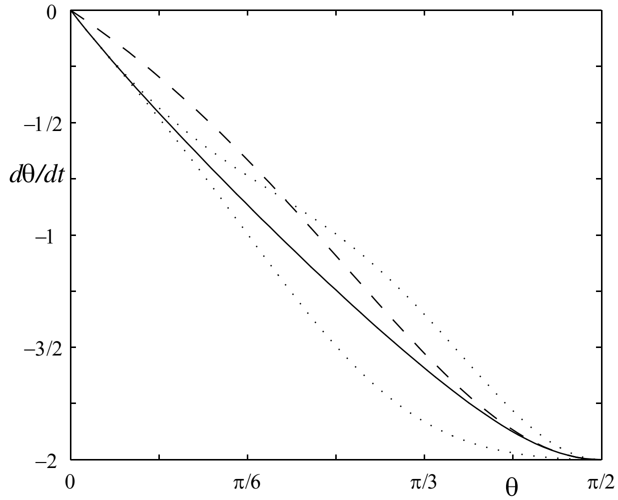


FIG. 2. Functional kinematic relationship of valvular opening, asymptotic curve (solid line), approximate asymptotic curve (dashed line), and arbitrary ±20% nonasymptotic curves (dotted line).

integration of the characteristic curve $\dot{\theta} = V(t)f(\theta)$ with a third order Runge-Kutta scheme; the reported solutions have been tested to be independent from the time discretization. The results from a series of simulations at different values of the Strouhal number are shown in Fig. 3. The value of the Strouhal number directly affects the ability of the flow to open a valve. When the Strouhal number is too large the leaflet is unable to complete its opening by the single pulse inflow; when it is small the valve opens very rapidly. For reference, the value of this Strouhal number for a mitral valve is about 0.1 and it is not much variable with the physiological frequency (remember that the period $T$ employed here is not the heartbeat period but the duration of the diastolic $E$ wave only). It is worth to point out that, when the leaflet is not massless, leaflet inertia will reduce the ability to follow rapid flow accelerations; its motion will present an initial slower acceleration and possibly higher, or longer lasting, velocities during flow deceleration.

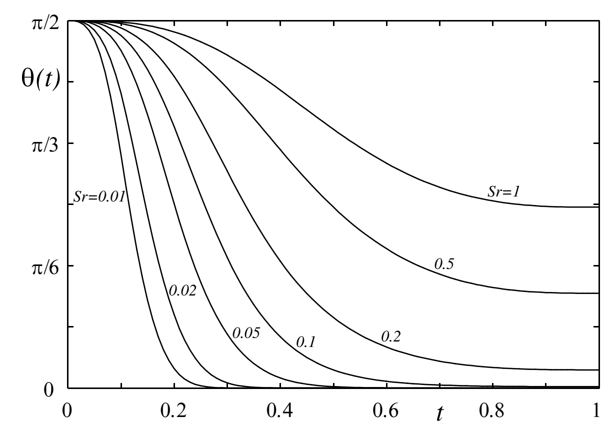


FIG. 3. Valvular opening profile for different values of the Strouhal number.

Before closing this subsection we wish to mention that the dashed curve in Fig. 2 represents the approximation

$$\frac{\dot{\theta}}{V} = \frac{2\sin\theta}{\sin\theta - 2}. \quad (3)$$

It is computed assuming that the average velocity on the open part of the channel, $(V + \dot{\theta}/2)/(1 - \sin\theta)$, gives a component round the edge equal to $-\dot{\theta}$; it is therefore an approximation of the Kutta condition. The approximation (3) presents the proper limiting behavior and otherwise underestimates the modulus of the angular velocity. In the present ideal model, the formula (3) has little value; such an approximate approach could be useful in complex conditions (possibly three dimensional) when a theoretical calculation is unpractical.

The characteristic leaflet curve, $\dot{\theta}/V = f(\theta)$, is a description of the asymptotic opening dynamics. In principle, this sole curve can be used to evaluate valvular opening. The profile of Fig. 2 was evaluated in the limit $\mathrm{Re} \to \infty$; its validity is now analyzed at a moderate value of the Reynolds number by the numerical solution of the two-dimensional Navier-Stokes equations. The flow starts from rest at $t = 0$ and the plate is subjected to a predetermined motion $\theta(t)$ corresponding to the profile $V(t)$.

The two-dimensional Navier-Stokes equations are written in the vorticity stream function, $(\omega, \psi)$, formulation [12,13]. The vorticity equation

$$\frac{\partial\omega}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} = (\mathrm{Sr}\,\mathrm{Re})^{-1}\nabla^2\omega, \quad (4)$$

with the Poisson relation $\nabla^2\psi = -\omega$, is solved on a discrete domain using second order centered finite differences [13]. The moving plate is reproduced by a $\omega$, $\psi$ version of the basic immersed boundary approach [14] where the grid points nearest to the plate are marked as boundary points. This approach, although less accurate about the immersed boundary, allows us to simulate the system for any valvular position including when it is closed, $\theta = \pi/2$, and open, $\theta = 0$. On the other side, it requires a high resolution to avoid the solution degrading about the immersed boundary; the results have been computed with a resolution $\Delta x = \Delta y = 1/128$ without appreciable differences from $1/64$. The time advancement is performed with a second order Crank-Nicholson method, implicit for the viscous term and Adam-Bashfort for the nonlinear term.

A Navier-Stokes simulation has been first performed in correspondence of the asymptotic valvular motion that automatically satisfies the Kutta condition in the irrotational model. The Strouhal number is $\mathrm{Sr} = 0.1$ and the Reynolds number has been set to $\mathrm{Re} = 10^3$. Results show that indeed no vortex shedding develops during the entire simulation and confirm the validity of the characteristic curve of Fig. 2 at moderate Re; one snapshot of the flow

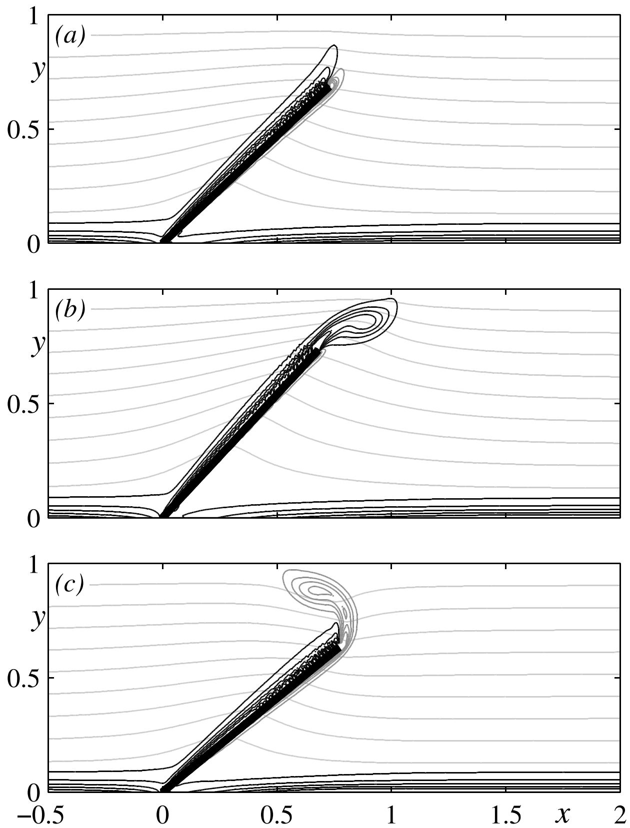


FIG. 4. Streamlines (light gray) and vorticity distribution at $t = 0.25$, for $\mathrm{Sr} = 0.1$, $\mathrm{Re} = 10^3$, corresponding to the asymptotic curve (a) and the $\pm 20\%$ nonasymptotic curves (b, c). Vorticity levels from $\pm 20$ to 180 step 40; positive (clockwise) values are black, while negative values are gray.

field is shown in Fig. 4(a), at $t = 0.25$. Two simulations have been run with an accelerated and decelerated valvular motion obtained by multiplying the theoretical angular velocity with the function $1 \pm 0.1(1 - \cos 4\theta)$ that gives a maximum difference of $\pm 20\%$ with respect to the theoretical value. This form has been arbitrarily chosen to have the same limiting behavior when $\theta = 0$, $\pi/2$, such that either the mass conservation is guaranteed when the valve is closed and the valve cannot pass beyond $\theta = 0$. These profiles are also shown, dotted, in Fig. 2. The results, in Figs. 4(b) and 4(c) for the slower and faster profile, respectively, demonstrate that when the leaflet moves slower than the asymptotic value, pressure is higher on the upstream face and a forward vortex shedding is found. On the opposite, when the valve is faster, a backward vortex shedding establishes.

The opening motion of the valvular leaflets should be derived by integration of the equation of motion for the solid structure, coupled in a strong interaction with the fluid evolution. This approach is commonly unpractical, because either the leaflet mechanical structure is not adequately described or material parameters are unknown. In addition, the modeling of a realistic fluid-structure interaction, that is of the strong type, is still challenging and many results (that employ extreme under-relaxed coupling) are questionable. We have introduced the asymptotic description of the valvular motion on the basis of its kinematics only. The asymptotic characteristic curve represents a reference bound to realistic dynamics either modeled or experimentally measured. On this basis, a real characteristic curve can be reconstructed using available data. It could represent the principal characterization of the valvular dynamics.

The work has been partially supported by Italian MIUR under the Grants No. FIRB 2001 RBAU01Z44F and No. PRIN 2004089599.

*Electronic address: giannip@dic.univ.trieste.it

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