

## Theory of Mesoscopic Magnetism in Photonic Crystals

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We provide a rigorous theoretical basis for the artificial magnetic activity of metamaterials near resonances. Our approach is a renormalization-based scheme that authorizes a completely general theory. The major result is an explicit expression of the effective permeability, in terms of resonant frequencies. The theoretical results are checked numerically, and we give applications of our theory to left-handed media and to the solution of the Pokrovski-Efros paradox.

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Photonic crystals are artificial devices, periodically structured, that exhibit photonic band gaps [1]. Dielectric photonic crystals are considered in the optical domain, but metallic ones (or wire mesh photonic crystals) are also studied in the microwave or terahertz range [2,3]. It has been well established [4] that, below a cut frequency, wire mesh photonic crystals behave as if they were homogeneous with a negative, frequency-dependent permittivity given by  $\varepsilon_{\text{eff}} = 1 - 2\pi\gamma/(\omega/c)^2$ , where  $1/\gamma = d^2 \log(d/r)$  (here  $d$  is the period of the crystal and  $r$  is the radius of the wires) [5]. For a frequency below  $\omega_p = \sqrt{2\pi\gamma}c$ , the homogenized permittivity is negative and the propagation of waves is forbidden. The homogenized permittivity represents the scattering behavior of the wire mesh photonic crystal for large enough wavelengths, and explains why these structures present a photonic band gap down to the null frequency (at least for infinitely conducting wires). Recently, Pendry and coauthors suggested that it was possible to design photonic crystals with nonmagnetic materials in such a way that they possess an artificial magnetic activity [6] and are described by an effective permeability. Basically, two geometries have been suggested: split ring resonators and dielectric fibers with a large permittivity [6,7]. It is believed that with these geometries it is possible to obtain a negative permeability and, by adding a wire mesh structure, to design a material with both a negative permittivity and a negative permeability. Materials with these characteristics do not seem to exist in nature, and therefore one tries to design them artificially (they are called “left-handed materials”). They were studied theoretically long ago in a speculative and quite fascinating work by Veselago [8]. He showed that they behave as if they had a negative index. Among other properties, the Snell-Descartes law is reversed: at the interface between air and the material, a beam is refracted on the same side of the normal. These ideas have motivated a lot of work, both experimentally and numerically (in particular, in [9]), as well as a lot of polemical issues [10].

It seems, however, that, hitherto, there is no unified theoretical approach to this kind of effective behavior. The method generally followed consists in characterizing the scattering matrix of a basic resonator by means of the  $\varepsilon$  and  $\mu$  parameters, and then deriving the effective parameters without taking into account the coupling between each resonator [11]. In the present work, we address this problem by using a renormalization group analysis, which gives us deep insight into the phenomena and predicts that the existence of a negative  $\mu$  is linked to internal resonances. In fact, the possibility of getting negative permeability is very different from the possibility of getting negative permittivity: while the negative  $\varepsilon$  is obtained for low frequencies (i.e., large wavelength with respect to the wires constituting the crystal), the negative  $\mu$  is obtained in the resonant domain, and only for a rather small interval of frequencies. In particular, our approach explains the apparent paradox, raised by Pokrovsky and Efros [12], that, by embedding wires in a medium with negative  $\mu$ , one does not get a left-handed medium. It also gives a complete analysis of the effective properties of a wire mesh photonic crystal, with very high conductivity. Before going into the details of our study, we stress that we have tried here to make a bridge between two domains that seem to be in opposition: that of the resonances and that of homogenization.

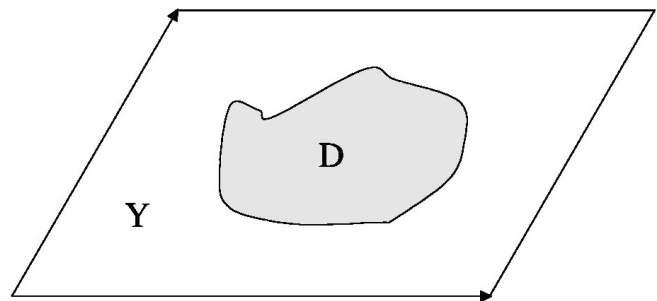


FIG. 1. The basic cell of the photonic crystal.

In the following, we consider a 2D photonic crystal whose Wigner-Seitz cell  $Y$  is given in Fig. 1. It is made of a dielectric rod (relative permittivity  $\varepsilon_i$ , cross section  $D$ ) embedded in a dielectric matrix  $\varepsilon_e$ . When the contrast between  $\varepsilon_i$  and  $\varepsilon_e$  is substantial enough, there appear Mie resonances into the highest index material. It was suggested some time ago [13] that such internal resonances might result in the opening of forbidden gaps.

Our point is to show that, near these resonances, the device behaves as if it had homogeneous electromagnetic parameters  $\varepsilon_h$  and  $\mu_h$ . Of course, for this situation to be physically sound, the resonant wavelengths should be larger than the period; otherwise the medium could not be described by homogeneous parameters. That is why we require, as in [6], that  $\varepsilon_i$  be much higher than  $\varepsilon_e$ . The method that we employ consists of changing (“renormalizing”) the properties of the medium while keeping the relevant physical phenomena, i.e., the resonances, unchanged. To do so, we choose a small number  $\tau < 1$ , and we proceed to the following operation, denoted by  $\mathcal{R}$ . We multiply the radius of the rods and the period by  $\tau$ , while maintaining constant the domain  $\Omega$  where the rods are contained (the number  $N$  of rods is increased as  $N \sim |\Omega|/\tau^2$ ). We divide the permittivity  $\varepsilon$  of the rods by  $\tau^2$  (the optical diameter remains constant). The wave is  $p$  polarized so that the induction field reads  $\mathbf{B}(\mathbf{x}) = u(\mathbf{x})\mathbf{e}_z$ , but the vectorial form will prove useful for the analysis. We write  $\mathbf{B}(\mathbf{x}; \mathcal{R})$  and  $\mathbf{E}(\mathbf{x}; \mathcal{R})$ , the fields scattered by the renormalized structure. The point is to iterate this operation  $n$  times and study the limit of  $\mathbf{B}(\mathbf{x}; \mathcal{R}^n)$  and  $\mathbf{E}(\mathbf{x}; \mathcal{R}^n)$  as  $n$  tends to infinity. In order to do so, we use a two-scale expansion of  $(\mathbf{E}, \mathbf{B})$ :

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_0(\mathbf{x}, \mathbf{x}/\tau^n) + \tau^n \mathbf{B}_1 + \dots, \\ \mathbf{E} &= \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\tau^n) + \tau^n \mathbf{E}_1 + \dots, \end{aligned} \quad (1)$$

where the fields  $\mathbf{E}_0, \mathbf{B}_0$  depend on both the real space variable  $\mathbf{x}$  (the global variable) and on the Wigner-Seitz cell variable  $\mathbf{y}$  (the local variable). The fields are periodic with respect to  $\mathbf{y}$ . Our point is to find the limit fields  $\mathbf{E}_0, \mathbf{B}_0$ . The local variable is in fact a hidden one: it is an internal degree of freedom. The true (observable) macroscopic fields  $(\mathbf{E}_h, \mathbf{B}_h)$  are the averages of the microscopic fields  $(\mathbf{E}_0, \mathbf{B}_0)$  over  $Y$ :

$$\mathbf{B}_h(\mathbf{x}) = \int_Y \mathbf{B}_0(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \quad \mathbf{E}_h(\mathbf{x}) = \int_Y \mathbf{E}_0(\mathbf{x}, \mathbf{y}) d\mathbf{y}. \quad (2)$$

Although we do not find it relevant to present all the mathematical details, we believe that it is important to offer the reader a general view of the method employed to get the limit fields. A complete and rigorous mathematical derivation can be found in [14]

We analyze first the behavior of the fields with respect to the local variable. That is, we wish to describe the microscopic behavior of the fields with respect to their internal degrees of freedom. Using the expansion (1) of the field,

the  $\nabla \times \cdot$  operator is transformed into

$$\nabla \times \cdot \rightarrow \nabla_x \times \cdot + \tau^{-n} \nabla_y \times \cdot$$

(we have to make explicit on which variables the derivations operate because there are two sets of variables). Plugging these expressions into the Maxwell system and identifying the terms that correspond to the same power of  $\tau^n$ , we obtain the following system for the microscopic electric field:

$$\nabla_y \times \mathbf{E}_0 = 0 \text{ on } Y, \quad \nabla_y \cdot \mathbf{E}_0 = 0 \text{ on } Y \setminus D \quad (3)$$

Besides,  $\mathbf{E}_0 = 0$  on  $D$  and  $\mathbf{E}_1 = 0$  on  $Y \setminus D$ . This system is of the electrostatic type:  $\mathbf{E}_0$  does not depend on the microscopic induction field, nor does it depend upon the wavelength. As a matter of fact, on  $Y \setminus D$ ,  $\mathbf{E}_0$  does not depend on the variable  $\mathbf{y}$ , as can be deduced from system (3). Let us now turn to the magnetic field. The system satisfied by  $\mathbf{B}_0$  is of an entirely different nature:

$$\nabla_y \times \mathbf{B}_0 = -i\omega \varepsilon_i \mathbf{E}_1 \text{ on } Y, \quad \nabla_y \times \mathbf{E}_1 = i\omega \mathbf{B}_0 \text{ on } D. \quad (4)$$

We have obtained a microscopic Maxwell system that describes the microscopic behavior of the fields. It can be seen that  $\mathbf{E}_1$  gives indeed a first order expansion of the field inside  $D$ : it replaces  $\mathbf{E}_0$ , which is null there. Let us now use the fact that the fields are polarized. We write  $\mathbf{B}_0(\mathbf{x}) = u_0(\mathbf{x})\mathbf{e}_z$ . Plugging this expression into system (4) shows that the magnetic field is independent of  $\mathbf{y}$  on  $Y \setminus D$ . Next, by combining the equation in system (4), we find that

$$\Delta_y u_0 + k^2 \varepsilon_i u_0 = 0 \text{ on } D, \quad u_0 = \text{const on } Y \setminus D. \quad (5)$$

From this system, we deduce that the microscopic induction field is linked to the macroscopic one by  $u_0(\mathbf{x}, \mathbf{y}) = [m(\mathbf{y})/\mu_h]u_h(\mathbf{x})$ , where  $m$  satisfies

$$\Delta_y m + k^2 \varepsilon_i m = 0 \text{ on } D, \quad m = 1 \text{ on } Y \setminus D, \quad (6)$$

and  $\mu_h$ , which shall be interpreted below as a relative permeability, is the mean value of  $m: \mu_h = \int_Y m(\mathbf{y}) d\mathbf{y}$ . Until now, we have clarified what happens at the microscopic scale. The point now is to derive the equations that are satisfied by the macroscopic fields. The propagation equations read, for  $\mathbf{y} \in Y \setminus D$ , as

$$\begin{aligned} \nabla_x \times \mathbf{B}_0 + \nabla_y \times \mathbf{B}_1 &= -i\omega \varepsilon_0 \varepsilon_e \mathbf{E}_0, \\ \nabla_x \times \mathbf{E}_0 &= i\omega \mathbf{B}_0. \end{aligned} \quad (7)$$

In the first line, we recognize the Maxwell-Ampère equation with the extra term  $\nabla_y \times \mathbf{B}_1$ . This term is homogeneous to an electric displacement field, and it represents the polarization due to the presence of the scatterers. Indeed, in the long wavelength regime, the emission diagram of a fiber is that of a dipole (for the  $p$  polarization). As a consequence, the whole set of fibers that constitutes the photonic crystal behaves as a set of coupled dipoles, producing a possibly anisotropic permittivity tensor. More precisely, since  $u_0$  is independent of  $\mathbf{y}$  on  $Y \setminus D$ , we obtain

the following system satisfied by  $u_1$ :

$$\Delta_y u_1 = 0 \text{ on } Y \setminus D, \quad \frac{\partial u_1}{\partial n} = -\mathbf{n} \cdot \nabla_x u_0 \text{ on } D, \quad (8)$$

where  $\mathbf{n} = (n_1, n_2)$  is the normal vector field to  $D$ . This system implies a linear relation between  $u_1$  and  $u_0$  of the form  $\nabla_y u_1 = \mathcal{P}(\mathbf{y}) \nabla_x u_0$ , where

$$\mathcal{P}(\mathbf{y}) = \begin{pmatrix} 1 + \frac{\partial w_1}{\partial x_1} & \frac{\partial w_2}{\partial x_1} \\ \frac{\partial w_1}{\partial x_2} & 1 + \frac{\partial w_2}{\partial x_2} \end{pmatrix}, \quad (9)$$

and  $w_i$  satisfies

$$\Delta w_i = 0 \text{ on } Y \setminus D, \quad \frac{\partial w_i}{\partial n} = -n_i \text{ on } \partial D. \quad (10)$$

It is not difficult to see [15,16] that  $A_h = \int_{Y \setminus D} \mathcal{P}(\mathbf{y}) d\mathbf{y}$  is the inverse of the effective permittivity tensor  $\varepsilon_h (= A_h^{-1})$  of the photonic crystal. The effective macroscopic equation can now be obtained by averaging system (7) on  $Y \setminus D$ :

$$\nabla \cdot [\varepsilon_h^{-1} \nabla (\mu_h^{-1} u_h)] + k^2 u_h = 0. \quad (11)$$

The macroscopic behavior of the system is characterized by an effective permittivity tensor  $\varepsilon_h$  and an effective permeability  $\mu_h$ . This shows that the system exhibits an artificial magnetic activity. There are two huge differences between the effective permittivity and the effective permeability: the permittivity can be a matrix, and hence the medium can be anisotropic, whereas the effective permeability is always a scalar, and therefore no anisotropic permeability can be obtained. Second, the permittivity is not frequency dependent; it is a static permittivity, whereas the permeability depends on the frequency. Let us give a closer look to the system of equations that defines the effective permeability  $\mu_h$ . As it stands in (6), it is just a partial differential equation problem. Under this form, its physical meaning does not appear clearly. To make it more explicit, let us recast it into an eigenvalue problem. This will help us to understand the underlying physics of what might look, at this stage, as a rather abstract result. The system (6) has a unique solution only if there is no function  $\psi$  such that  $\psi$  is null on  $Y \setminus D$  and  $\psi$  satisfies the same Helmholtz equation on  $D$ . Otherwise,  $m + \psi$  would still be a solution of (6). Following spectral theory [17], we denote  $H = -\varepsilon_i^{-1} \Delta$  and we look for functions  $\Phi$  satisfying the eigenvalue problem:

$$\Phi = 0 \text{ on } Y \setminus D, \quad H\Phi = k^2 \Phi \text{ on } D. \quad (12)$$

We get a set of eigenvalues  $k_n^2$  and a set of eigenfunctions  $|\Phi_n\rangle$ . The physical meaning of these eigenvalues can be understood by going back to the unrenormalized initial fiber, with permittivity  $\varepsilon_i$ . This fiber alone possesses resonant frequencies. They correspond to modes that are strongly localized inside the fiber. However, when there is a large number of fibers, these resonances are slightly shifted due to coupling, and furthermore, these resonances are modified by the renormalization process. That is pre-

cisely why the eigenvalues of problem (12) are the renormalized Mie frequencies of the fiber.

For a given wave vector  $k^2$ , we look for a solution  $m$  by expanding it on the basis  $|\Phi_n\rangle$ , by noting that  $m - 1$  is null on  $Y \setminus D$ :  $m(\mathbf{y}) = 1 + \sum_n m_n |\Phi_n\rangle$ . The coefficients are obtained by inserting this expansion into (6). We get, after averaging, the effective permeability  $\mu_h = \langle 1|m \rangle$  under the form

$$\mu_h(k) = 1 + k^2 \sum_n \frac{|\langle \Phi_n | 1 \rangle|^2}{k_n^2 - k^2}. \quad (13)$$

We have obtained a completely general expression for the effective permeability. It relies on the cavity modes of the fiber only. In the vicinity of a resonance  $k_n^2$ , we have  $\mu_h \sim 1 - k_n^2 |\langle \Phi_n | 1 \rangle|^2 (k^2 - k_n^2)^{-1}$ , which shows, in complete generality, that the permeability exhibits anomalous dispersion near the resonances, and becomes negative there. It should also be noted that only the eigenfunctions with nonzero mean value contribute. This is due to the fact that we have kept only the first order terms in the expansions (1).

Let us give an explicit computation in case of a square fiber. The derivation is rather straightforward and follows closely that of the well-known  $TE$  modes in waveguides with square section. The eigenfunctions are  $\Phi_{nm}(\mathbf{y}) = 2 \sin(n\pi y_1) \sin(m\pi y_2)$ , and the corresponding eigenvalues are  $k_{nm}^2 = \pi^2(n^2 + m^2)$ . The expansion of  $m$  on this basis leads to the following effective permeability:

$$\mu_h(k) = 1 + \frac{64a^2}{\pi^4} \sum_{(n,m)\text{odd}} \frac{k^2}{n^2 m^2 (k_{nm}^2 - k^2)}, \quad (14)$$

where  $\tilde{k}_{nm}^2 = k_{nm}^2 / a^2 \varepsilon_i$ . Let us now turn to some numerical applications. First, we note that our analysis is supposed to

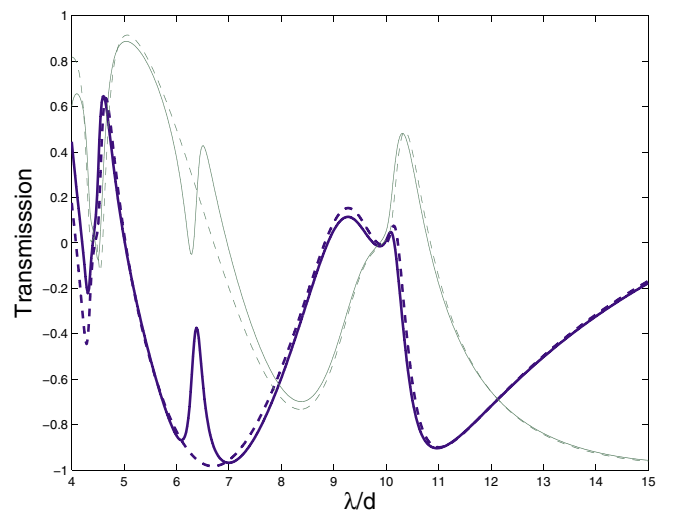


FIG. 2 (color online). Real (thick lines) and imaginary (thin lines) parts of the transmission for the metamaterial (solid lines) and the homogenized material (dashed lines).

work when there are Mie resonances at wavelengths large with respect to the period of the crystal. This was the situation described in [7], where  $\varepsilon = 200 + 5i$ . We choose this value for our own numerical computations, the point being to test the validity of the theory.

Using a rigorous diffraction code for gratings [18], we plot the transmission spectrum (dashed lines in Fig. 2) for a stack of three diffraction gratings made of square rods. There is a band gap for  $\lambda/d$  between 8 and 12, due to a Mie resonance. In order to test our results, we plot the transmission spectrum of a homogeneous slab (solid lines in Fig. 2) with parameters  $\varepsilon_h = 1.7$  [this value is obtained numerically from the resolution of problem (10)] and  $\mu_h$  given in (14). We see in Fig. 2 that both curves fit very well, indicating that, although the wavelength is not that large, the whole photonic crystal behaves as a homogeneous magnetic material. The discrepancy that is seen around  $\lambda/d = 6.5$  is due to the presence of a Mie resonance of null mean value that is not taken into account in our theory. Only by expanding the fields to the second order could we incorporate this resonance in our global result.

Finally, let us use these results to analyze recent problem. In [12], Pokrovsky and Efros showed that it was not possible to design a negative index medium by embedding metallic wires into a matrix with a negative  $\mu$ , whereas the converse is possible. This can be explained in the following manner: The negative permittivity is obtained as a macroscopic effect, by which we mean that it is an interference effect and not an effect that takes place at the scale of the microscopic cell only. In a much different way, the negative permeability is obtained as a purely local effect, which happens at the scale of the microscopic cell. Therefore, for this effect to occur, no strong coupling between the fibers is requested; the coupling has to be sufficient just enough so that the incident field can reach the fibers by the tunnel effect. In our model, the propagation equation of the structure is obtained immediately by replacing  $\varepsilon_e$  by  $-\varepsilon_e$ . Then, near the regions of anomalous dispersion, both parameters are negative and the propagation equation is the usual Helmholtz equation. Consequently, the field can propagate. On the contrary, for metallic wires in a medium with negative  $\mu$ , the propagation equation reads [16,19]  $\Delta u + k^2 \mu(1 - 2\pi\gamma/\mu k^2)u = 0$ , which leads to evanescent waves.

We have given in this work a theory of the mesoscopic magnetism in metamaterials. We have shown that it is possible to give a homogenized description of a heterogeneous device in the resonance domain. To do so, we have used a renormalization approach that shows that two scales should be distinguished: a microscopic one and a macroscopic one. We have shown that the artificial, mesoscopic magnetism is due to microscopic magnetic moments in-

duced by geometric resonances. We have considered here a two-dimensional photonic crystal, which allows one to study polarized fields. Work is in progress to extend the approach to structures of finite extension, i.e., fibers of finite length. New phenomena are to be expected due to the coupling between the  $E$  and  $H$  fields. However, the effective magnetic behavior still remains, and for in-plane waves, i.e., for an incident field polarized along the axis of the fibers, the theory presented here is valid. This situation is the one encountered in the published experimental studies. So far, the analysis works for high permittivities, but we stress that there are inner resonances in gratings for much lower contrasts as well [20]. Therefore, we do believe that the same physics can be found in the optical range of wavelengths.

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