Mechanism of Stabilization of Ballooning Modes by Toroidal Rotation Shear in Tokamaks

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A ballooning perturbation in a toroidally rotating tokamak is expanded by square-integrable eigenfunctions of an eigenvalue problem associated with ballooning modes in a static plasma. Especially a weight function is chosen such that the eigenvalue problem has only the discrete spectrum. The eigenvalues evolve in time owing to toroidal rotation shear, resulting in a countably infinite number of crossings among them. The crossings cause energy transfer from an unstable mode to the infinite number of stable modes; such transfer works as the stabilization mechanism of the ballooning mode.

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The theory of ideal magnetohydrodynamic (MHD) ballooning modes in a static plasma has been well established by applying the Wentzel-Kramars-Brillouin (WKB) method [1,2]. The WKB method was also applied for the ideal MHD ballooning modes in a toroidally rotating tokamak, and coupled wave equations along a magnetic field line were derived [3]. The behavior of the solution has not been fully understood, although several numerical solutions have been obtained [4–8]. In Ref. [7], it was found that toroidal rotation shear damps the perturbation energy of ballooning modes; the damping phase alternates with an exponentially growing phase. When the damping compensates the growth in the sense of time average, the plasma is marginally stable. However, it has not been clarified how the toroidal rotation shear damps the perturbation energy.

In the present Letter, we explore how toroidal rotation shear stabilizes ballooning modes via a spectral analysis of a linear differential operator for ballooning modes in a static plasma. If we crudely try to expand the ballooning perturbation in a rotating plasma by a set of eigenfunctions for ballooning modes in a static plasma, such an attempt will fail because of difficulty in treating the continuous spectrum; the generalized eigenfunctions belonging to the continuous spectrum are singular and non-square-integrable. Thus we cannot treat them numerically, and also analytical studies are limited by geometry, profiles, and so on. An earlier work tried to resolve the difficulty by replacing it with the closely spaced discrete spectrum [9], which is accomplished by replacing the covering space with a large but finite interval. However, when the ballooning mode in a rotating tokamak propagates in a wide region, we have to take the interval large enough. Then the spacing of the eigenvalues become closer and the eigenfunctions become singular, which indicates the difficulty in the analysis. We have formulated an associated eigenvalue problem yielding only the discrete spectrum in the original covering space [10]; it is composed of the linear differential operator in a static plasma and a weight function which is chosen to generate only the discrete spectrum. Then, we obtain a complete set of square-integrable eigenfunctions which is defined in the same domain as the ballooning mode in a toroidally rotating tokamak, which enables us to expand the mode.

Actually, the toroidal rotation shear changes the wave vector in time [3], and the coefficients of the ballooning equation depend on time only through the wave vector. As we see in the following, the ballooning equation is composed of time derivatives of the plasma displacement and a space-derivative operator (the ballooning operator). The ballooning operator includes time just as a parameter; thus we can define an eigenvalue problem associated with the ballooning operator at each instance. Then we can expand the ballooning perturbation by the eigenfunctions which includes time as a parameter, or which varies in time. The eigenvalues also vary in time. As we find below, countably infinite numbers of crossings of the eigenvalues occur around the time when the smallest eigenvalue changes its sign. The crossings cause energy transfer from an unstable mode to the infinite number of stable modes, and such transfer stabilizes the mode.

The ballooning equations in toroidally rotating tokamaks were derived as coupled wave equations for two components of a displacement vector [3]. Features of the equations are as follows: (i) convection terms exist and (ii) the coefficients of the equations have the dynamical lattice symmetry [3]. We proposed a model equation which has the above two features in Ref. [11],

$$\bar{\rho}\left(\frac{\partial^2 \xi_\perp}{\partial t^2} - U\frac{\partial \xi_\perp}{\partial t}\right) = \mathcal{L}\xi_\perp, \tag{1}$$

where

$$\mathcal{L}\xi_\perp \equiv \frac{\partial}{\partial \vartheta}\left(f\frac{\partial \xi_\perp}{\partial \vartheta}\right) - g\xi_\perp, \tag{2}$$

$$\bar{\rho} \equiv \frac{\mu_0 \rho |\mathbf{k}|^2 \sqrt{g}}{B^2}, \tag{3}$$

$$U \equiv \frac{2\mathbf{k}\cdot\nabla\Omega}{|\mathbf{k}|^2}, \tag{4}$$

$$f \equiv \frac{|\mathbf{k}|^2}{B^2\sqrt{g}}, \tag{5}$$

$$g \equiv -\frac{2\mu_0}{B^4}(\mathbf{B}\times\mathbf{k}\cdot\boldsymbol{\kappa})(\mathbf{B}\times\mathbf{k}\cdot\nabla p). \tag{6}$$

It should be noted that Eq. (1) reduces to the conventional ballooning equation in a static plasma when the toroidal rotation shear Ω' is set to zero. Here, $\xi_\perp$ is a perpendicular component of the displacement vector, μ_0, ρ, p and Ω are vacuum permeability, mass density, pressure, and toroidal rotation frequency, respectively, $\mathbf{B}$ is a magnetic field, $\boldsymbol{\kappa}$ is a magnetic curvature, $\mathbf{k} \equiv \nabla\zeta - q\nabla\theta - (\vartheta - \theta_k + \dot{\Omega}t)\nabla q$ is a wave vector, θ and ζ are a poloidal angle and a toroidal angle, respectively, ϑ is an extended poloidal angle in the covering space, $\sqrt{g}$ is the Jacobian, θ_k is a ballooning angle, q is a safety factor, and $\dot{\Omega}$ is defined by $d\Omega/dq$. We have confirmed that damping phases appear in the time evolution of $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ by solving Eq. (1) numerically [11]. The damping phase alternates with an exponentially growing phase, which is essential for the stabilization of the ballooning modes [7]. As an example, time evolution of $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ is shown in Fig. 1 for $\dot{\Omega}\tau_A = 0$ (no rotation shear), 0.118 (unstable), and 0.473 (stable), where τ_A is the Alfvén time defined by (connection length)/(Alfvén velocity). Equation (1) is solved on a magnetic surface of a large aspect ratio and circular cross section tokamak equilibrium, and the magnetic shear and the pressure gradient parameters [12] on that surface are both chosen to be 2. The rotation shear is given arbitrarily on that surface without affecting its force balance (or equilibrium magnetic geometry) since we have chosen the rotation itself to be zero.

Here we expand $\xi_\perp$ in Eq. (1) by a set of orthonormal functions. If we take the set as eigenfunctions of the ballooning equation in a static plasma,

$$\mathcal{L}\xi = -\bar{\rho}\omega^2\xi, \tag{7}$$

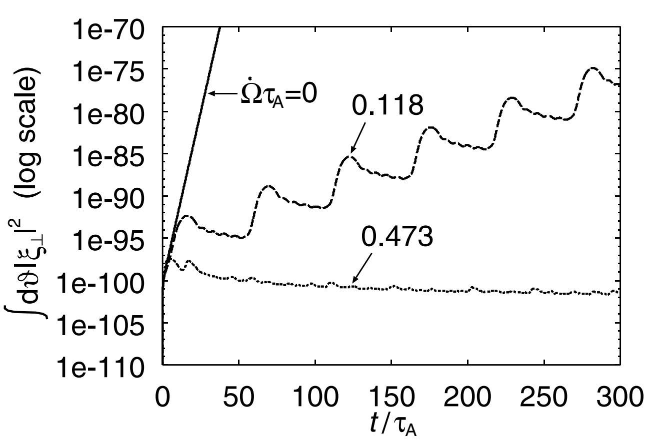

FIG. 1. Time evolution of $\int_{-\infty}^{\infty} d\vartheta|\xi_\perp|^2$ is shown for $\dot{\Omega}\tau_A = 0$ (no rotation shear), 0.118 (unstable), and 0.473 (stable).

we face intractable difficulty because of the continuous spectrum; the generalized eigenfunction corresponding to the continuous spectrum is singular and non-square-integrable which cannot be treated numerically. However, the spectrum of the linear operator $\mathcal{L}$ is determined by the operator itself together with a boundary condition and a weight function. If we appropriately choose the weight function, the spectrum becomes discrete [13]. In the present Letter, we devise the set of orthonormal functions as eigenfunctions of an associated eigenvalue problem

$$\mathcal{L}\xi = -w\lambda\xi, \qquad w \equiv h\bar{\rho}. \tag{8}$$

We have found that asymptotic solutions of ξ are proportional to $\vartheta^{-1/2\pm\mu}$ for any λ when $h \propto |\vartheta|^{-4}$ for large $|\vartheta|$ and $h = 1$ elsewhere [10]. Here $\mu \equiv \sqrt{1/4 - D_M}$ and D_M is the Mercier index: $D_M < 1/4$ is assumed which is usually satisfied in tokamaks. Then μ is real and positive. When the solution of Eq. (8) behaves as the large solution, $\lim_{\vartheta\to\infty}\xi \sim \vartheta^{-1/2+\mu}$, $\mathcal{L}$ is not self-ajoint even if $\lim_{\vartheta\to\infty}\xi = 0$ for $\mu < 1/2$. Only the small solution, proportional to $\vartheta^{-1/2-\mu}$, is acceptable. The norm $\int_{-\infty}^{\infty} d\vartheta w|\xi|^2$ is bounded for the small solution. Therefore, Eq. (8) has only the discrete spectrum. Since $\mathcal{L}$ is self-ajoint, the eigenfunctions are orthogonal with each other,

$$\int_{-\infty}^{\infty} d\vartheta w\xi_j\xi_k^* = \delta_{jk}, \tag{9}$$

where δ_{jk} is the Kronecker's delta function. We can hence expand $\xi_\perp$ as

$$\xi_\perp(t,\vartheta) = \sum_j a_j(t)\xi_j(t,\vartheta), \tag{10}$$

where a_j's are amplitudes and ξ_j's are eigenfunctions of

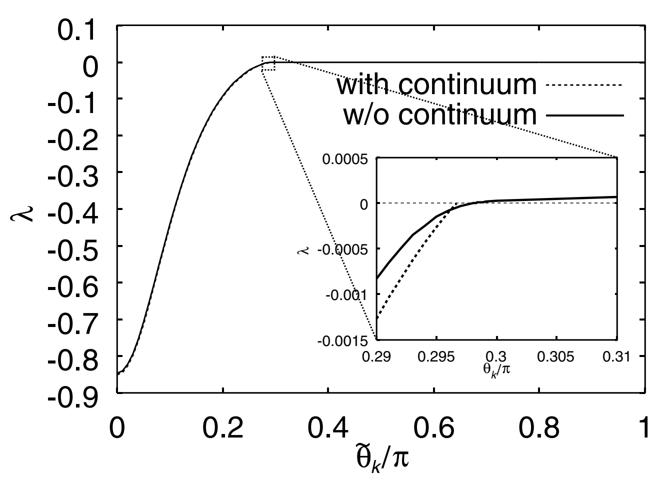

FIG. 2. The smallest eigenvalues of Eqs. (7) and (8) as functions of $\tilde{\theta}_k$. For $|\tilde{\theta}_k/\pi| < 0.297$, both Eqs. (7) and (8) yield the discrete eigenvalues. The eigenvalues nearly coincide. For $|\tilde{\theta}_k/\pi| > 0.297$, only Eq. (8) yields the discrete eigenvalue.

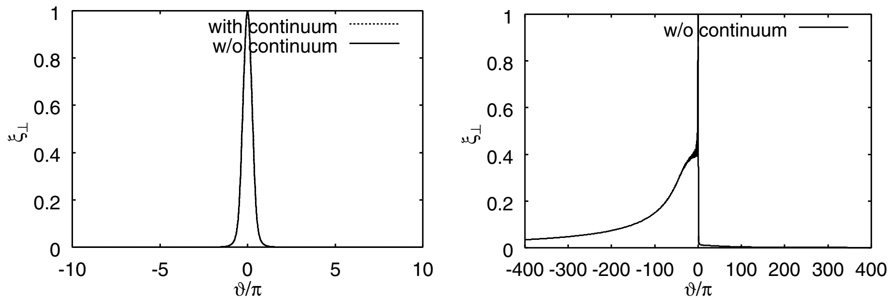

FIG. 3. Eigenfunctions corresponding to the smallest eigenvalues of Eqs. (7) and (8) as functions of ϑ. For $\tilde{\theta}_k/\pi = 0$ (left), the eigenfunctions are well localized in ϑ space and both curves are indistinguishable. For $\tilde{\theta}_k/\pi = 0.5$ (right), only Eq. (8) yields the square-integrable eigenfunction.

Eq. (8) with the ballooning angle $\tilde{\theta}_k \equiv \theta_k - \dot{\Omega} t$ taken to be a constant; let us note that the eigenvalues and the eigenfunctions are defined at each instance. It is noted that the discrete eigenvalues of Eq. (8) do not mean the spectral resolution of the Alfvén wave. Also, they are not a subset of the spectrum of Eq. (7). However, a small number of the eigenfunctions can approximate dominant behavior of the ballooning mode, and also the eigenfunctions form a complete set which expands any square-integrable function [10].

Figure 2 shows the smallest eigenvalues of both eigenvalue problems (7) and (8). When we solve Eq. (8) numerically, $h = 1$ and $h \propto |\vartheta|^{-4}$ are smoothly connected at $\vartheta = \pm 40\pi$ in this study. Although it is possible to take this connection point much farther from the origin, we have to take a much wider region to achieve good convergence of the eigenvalues and eigenfunctions. Since w is positive everywhere, the marginally stable state of Eq. (8) is the same as that of Eq. (7). Equation (8) yields the discrete eigenvalue for every $\tilde{\theta}_k/\pi$, although Eq. (7) does only when the eigenvalue is negative. Figure 3 shows the eigenfunctions corresponding to the smallest eigenvalues for $\tilde{\theta}_k/\pi = 0$ and 0.5. For $\tilde{\theta}_k/\pi = 0$, both Eqs. (7) and (8) yield the square-integrable eigenfunctions; they are localized in ϑ space and almost coincide since $h = 1$ except for $|\vartheta| \gg 1$. The corresponding eigenvalues also nearly coincide (see Fig. 2). For $\tilde{\theta}_k/\pi = 0.5$, on the other hand, only Eq. (8) yields the discrete eigenvalue and the square-integrable eigenfunction.

Substituting Eq. (10) into Eq. (1) and using the orthogonality condition (9), we obtain coupled evolution equations for a_j's as

$$\frac{d^2 a_j}{dt^2} + \sum_k C_{1jk} \frac{da_k}{dt} + \sum_k C_{2jk} a_k = -\sum_k C_{3jk} \lambda_k a_k, \quad (11)$$

where the coupling parameters are defined as

$$C_{1jk} \equiv \int_{-\infty}^{\infty} d\vartheta w \xi_k^* \Big(2\frac{\partial \xi_j}{\partial t} - U \xi_j \Big), \quad (12)$$

$$C_{2jk} \equiv \int_{-\infty}^{\infty} d\vartheta w \xi_k^* \Big(\frac{\partial^2 \xi_j}{\partial t^2} - U \frac{\partial \xi_j}{\partial t} \Big), \quad (13)$$

$$C_{3jk} \equiv \int_{-\infty}^{\infty} d\vartheta w \xi_k^* h \xi_j. \quad (14)$$

The coefficients C_{1jk} and C_{2jk} come both from the convection term U and from the time dependence of the wave vector. When $\dot{\Omega} = 0$, C_{1jk} and C_{2jk} vanish. In principle, we can reconstruct the solution of Eq. (1) from the solution of Eq. (11). However, it is impossible to solve an infinite number of coupled equations. When we solve the truncated

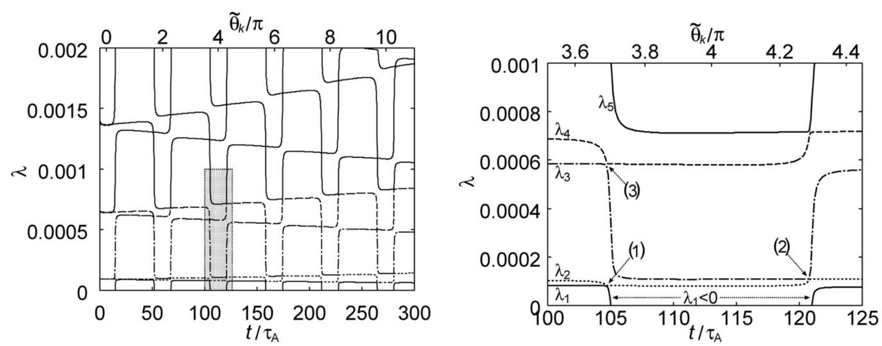

FIG. 4. Eigenvalues λ_j's as functions of t or $\tilde{\theta}_k$. Around the time when the smallest eigenvalue λ_1 changes its sign, countably infinite numbers of crossings of eigenvalues occur. The shaded region in the left is enlarged in the right.

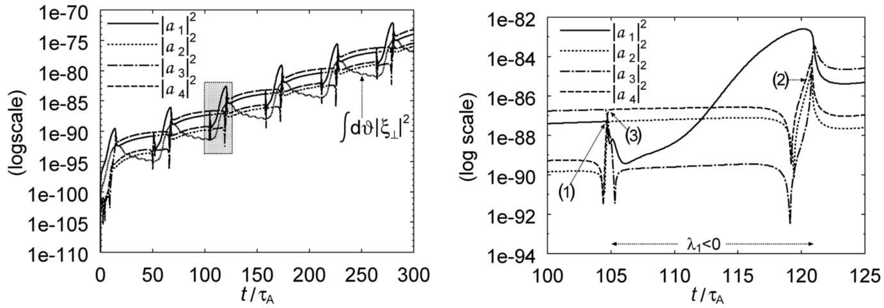

FIG. 5. Time evolution of $|a_j|^2$'s and $\int_{-\infty}^{\infty} d\vartheta |\xi_\perp|^2$ obtained from numerical solution of Eq. (1). When λ's cross (see Fig. 4), $|a_j|^2$ is smoothly converted to $|a_{j+1}|^2$. The shaded region in the left is enlarged in the right.

version of Eq. (11), we obtain an approximated solution. Such solutions will be presented in another Letter. Instead, we will expand the solution of Eq. (1) by the eigenfunctions of Eq. (8) and obtain a_j's in the following.

The expansion of $\xi_\perp$ by the square-integrable eigenfunctions enables us to clarify the stabilization mechanism. In the following, $\dot{\Omega}\tau_A$ is chosen to be 0.118, since we can observe the stabilization mechanism most clearly before complete stabilization. When $\dot{\Omega}$ is finite, $\tilde{\theta}_k$ changes with t in Eq. (8). Accordingly, the eigenvalues of Eq. (8) are functions of $\tilde{\theta}_k$ or t as shown in Fig. 4. We find that the toroidal rotation shear yields crossings of eigenvalues; from the numerical results, we see that a countably infinite number of crossings seem to occur at the same time. The periodic behavior of the eigenvalues comes from the dynamical lattice symmetry of the wave vector, which originates in the periodicity of a torus. The smallest eigenvalue, λ_1, becomes negative around $\tilde{\theta}_k = 2\pi m$ ($m = 0, \pm 1, \cdots$). Figure 5 shows the time evolution of $|a_j|^2$'s and $\int_{-\infty}^{\infty} d\vartheta |\xi_\perp|^2$ obtained from the numerical solution of Eq. (1). As shown in Fig. 5 (left), $\int_{-\infty}^{\infty} d\vartheta |\xi_\perp|^2$ grows during λ_1 is negative. The growth rate of $\int_{-\infty}^{\infty} d\vartheta |\xi_\perp|^2$ is almost the same with that of $|a_1|^2$. Thus the set of eigenfunctions of Eq. (8) nicely captures the behavior of the solution of Eq. (1). Around the time when λ_1 changes its sign, the eigenvalues cross as indicated by (1), (2), and (3). At (1), λ_1 and λ_2 cross. At that time, $|a_1|^2$ is smoothly converted to $|a_2|^2$ as indicated by (1) in Fig. 5. The mode couples through the convection term in Eq. (1), and the energy is transferred when the eigenvalues cross. As time proceeds, λ_2 and λ_3 cross [(2) in Fig. 4]. At that time, $|a_2|^2$ is converted to $|a_3|^2$ [(2) in Fig. 5]. Thereby, $|a_1|^2$ is converted to $|a_3|^2$ during $100 < t/\tau_A < 125$. As time proceeds further, λ_3 and λ_4 cross at $t/\tau_A \simeq 155$ and $|a_3|^2$ is converted to $|a_4|^2$ at that time [as same as (3) in Figs. 4 and 5]. Therefore, even if the unstable mode grows during λ_1 is negative, its energy is transferred to the infinite number of stable modes successively. For large $\dot{\Omega}$, the coupling among a_j's becomes strong, and the energy transfer completely stabilizes the ballooning mode. Other interesting behavior will be studied elsewhere.

In conclusion, we have succeeded in clarifying how the toroidal rotation shear stabilizes the ballooning mode. The toroidal rotation shear yields the crossings among the eigenvalues; the crossings cause energy transfer from the unstable mode to the countably infinite number of stable modes because of the mode couplings due to the convection. This mechanism can be interpreted as phase mixing which damps the unstable mode connected to the stable continuum through the toroidal rotation shear.

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