Topologically Protected Qubits from a Possible Non-Abelian Fractional Quantum Hall State

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The Pfaffian state is an attractive candidate for the observed quantized Hall plateau at a Landau-level filling fraction $\nu = 5/2$. This is particularly intriguing because this state has unusual topological properties, including quasiparticle excitations with non-Abelian braiding statistics. In order to determine the nature of the $\nu = 5/2$ state, one must measure the quasiparticle braiding statistics. Here, we propose an experiment which can simultaneously determine the braiding statistics of quasiparticle excitations and, if they prove to be non-Abelian, produce a topologically protected qubit on which a logical NOT operation is performed by quasiparticle braiding. Using the measured excitation gap at $\nu = 5/2$, we estimate the error rate to be 10^{-30} or lower.

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Introduction.—The computational power of a quantum-mechanical Hilbert space is potentially far greater than that of any classical device [1,2]. However, it is difficult to harness it because much of the quantum information contained in a system is encoded in phase relations which one might expect to be easily destroyed by its interactions with the outside world (“decoherence” or “error”). Therefore, error-correction is particularly important [3,4].

An interesting analogy with topology suggests itself: local geometry is a redundant way of encoding topology. Slightly denting or stretching a surface such as a torus does not change its genus, and small punctures can be easily repaired to keep the topology unchanged. Remarkably, there are states of matter for which this is more than just an analogy. A system with many microscopic degrees of freedom can have ground states whose degeneracy is determined by the topology of the system. The excitations of such a system have exotic braiding statistics, which is a topological effective interaction between them [5]. Such a system is said to be in a topological phase [6,7]. The unusual characteristics of quasiparticles in such states can lead to remarkable physical properties, such as a fractional quantized Hall conductance [8]. Such states also have intrinsic fault tolerance [9]. Since the ground states are sensitive only to the topology of the system, local interactions with the environment cannot cause transitions between ground states unless the environment supplies enough energy to create excitations which can migrate across the system and affect its topology.

A different problem now arises: if the quantum information is so well protected from the outside world, then how can we—presumably part of the outside world—manipulate it to perform a computation? The answer is that we must manipulate the topology of the system. In this regard, an important distinction must be made between different types of topological phases. In the case of those states which are Abelian, we can only alter the phase of the state by braiding quasiparticles. In the non-Abelian case, however, there will be a set of $g > 1$ degenerate states, $\psi_a, a = 1, 2, \ldots, g$ of particles at $x_1, x_2, \ldots, x_n$. Exchanging particles one and two might do more than just change the phase of the wave function. It might rotate it into a different one in the space spanned by the ψ_as:

$$\psi_a \to M^{12}_{ab}\psi_b. \tag{1}$$

On the other hand, exchanging particles two and three leads to $\psi_a \to M^{23}_{ab}\psi_b$. If M^{12}_{ab} and M^{23}_{ab} do not commute (for at least some pairs of particles), then the particles obey *non-Abelian braiding statistics*. In the case of a large class of states, the repeated application of braiding transformations M^{ij}_{ab} allows one to approximate any desired unitary transformation to arbitrary accuracy and, in this sense, they are universal quantum computers [10]. Unfortunately, no non-Abelian topological states have been unambiguously identified so far. Some proposals have been put forward for how such states might arise in highly frustrated magnets [11,12], but the best prospects in the short run are in quantum Hall systems, where Abelian topological phases are already known to exist. The best candidate is the quantized Hall plateau with $\sigma_{xy} = \frac{5}{2}\frac{e^2}{h}$. The 5/2 fractional quantum Hall state is now routinely observed [13] in high-quality (i.e., low-disorder) samples. In addition, extensive numerical work [14] using finite-size diagonalization and wave function overlap calculations indicates that the 5/2 state belongs to the non-Abelian topological phase characterized by a Pfaffian quantum Hall wave function [15,16]. The set of transformations generated by braiding quasiparticle excitations in the Pfaffian state is not computationally universal, but other non-Abelian states in the same family are computationally universal. Thus, it is important to (a) determine if the $\nu = 5/2$ state is, indeed, in the Pfaffian universality class and, if so, to (b) use it to store and manipulate quantum information. In this Letter, we propose an experimental device which can address both of these. Features of our device are inspired by antidot experiments measuring the charge of quasiparticles [17] in Abelian fractional quantum Hall states such as $\nu = 1/3$ and proposals for measuring their statistics [18]. Our measurement procedure relies upon quantum interference as in the electronic Mach-Zehnder interferometer in which Aharonov-Bohm oscillations were observed in a two-dimensional electron gas [19].

In order to establish which topological phase the $\nu = 5/2$ plateau is in, one must directly measure quasiparticle braiding statistics. Remarkably, this has never been done even in the case of the usual $\nu = 1/3$ quantum Hall plateau. Part of the problem is that it is difficult to disentangle the phase associated with braiding from the phase which charged particles accumulate in a magnetic field [18]. Ironically, it may actually be easier to measure the effect of non-Abelian braiding statistics because it is not just a phase and is therefore qualitatively different from the effect of the magnetic field.

Pfaffian facts.—To make this latter point clear, let us summarize some important properties of quasiparticles in the Pfaffian state. The Pfaffian state may be viewed as a quantum Hall state of p-wave paired fermions. The quasiparticles in this phase have charge $e/4$ (not $e/2$, as one might naively assume from the Landau-level filling fraction $\nu = 2 + \frac{1}{2}$; this emphasizes the importance of an experiment such as [17] to measure the quasiparticle charge at $\nu = 5/2$). When there are $2n$ quasiparticles at fixed positions in the system, there is a 2^{n-1}-dimensional degenerate space of states. Exchanging and braiding quasiparticles is related to the action of the $2n$-dimensional Clifford algebra on this space [20], as has recently been confirmed by direct numerical evaluation of the Berry matrices [21]. In particular, two charge-$e/4$ quasiparticles can "fuse" to form a charge-$e/2$ quasiparticle either with or without a neutral fermion in its core. One may view the charge-$e/2$ quasiparticle as the quantum Hall incarnation of a superconducting vortex with a fermionic zero mode in its core [22–25]. We will regard the presence or absence of a neutral fermion in this core state if the two charge-$e/4$ quasiparticles were fused as our qubit. So long as the two quasiparticles are kept far apart, the neutral fermion is not localized anywhere and, therefore, the qubit is unmeasurable by any local probe or environment. However, we can measure the qubit by encircling it with a charge-$e/4$ quasiparticle. The presence of the neutral fermion causes the state to acquire an extra factor of -1 during this process. The qubit can also be manipulated by taking another charge-$e/4$ quasiparticle between the two charge-$e/4$ quasiparticles comprising the qubit, i.e., around one but not the other. Such a process transforms a state without a neutral fermion into a state with one and vice versa. Thus, it flips the qubit (and also multiplies by i). By performing an experiment which measures this qubit, flips it, and then remeasures it, we can demonstrate that the $\nu = 5/2$ state is in a non-Abelian topological phase. Such an experiment can only work if the environment does not flip the qubit before we have a chance to measure it, so the success of this experiment would demonstrate the stability of a topological qubit in a non-Abelian quantum Hall state.

The claimed quasiparticle braiding properties can be seen from the form of the four-quasihole wave functions given in [20]. The ground state (g.s.) wave function takes the form [15,16]

$$\Psi_{\text{g.s.}}(z_j) = \prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \text{Pf}\left(\frac{1}{z_j - z_k}\right). \quad (2)$$

Where the Pfaffian (Pf) is the square root of the determinant of an antisymmetric matrix. If we write,

$$\Psi_{(13)(24)}(z_j) = \prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \text{Pf}\left(\frac{(z_j - \eta_1)(z_j - \eta_3)(z_k - \eta_2)(z_k - \eta_4) + (j \leftrightarrow k)}{z_j - z_k}\right) \quad (3)$$

then the four-quasihole wave functions can be written in a basis in which their braiding is completely explicit;

$$\Psi^{(0,1)}(z_j) = \frac{(\eta_{13}\eta_{24})^{1/4}}{(1 \pm \sqrt{x})^{1/2}}(\Psi_{(13)(24)} \pm \sqrt{x}\Psi_{(14)(23)}), \quad (4)$$

where $\Psi_{(14)(23)}$ is defined similarly to $\Psi_{(13)(24)}$, $\eta_{13} = \eta_1 - \eta_3$, etc., and $x = \eta_{14}\eta_{23}/\eta_{13}\eta_{24}$. The effect of braiding quasiparticles is a combination of the explicit monodromy of the wave function (4) and the Berry matrices obtained from adiabatic transport of the η_is; the phase factors in (4) have been chosen so that the latter are trivial and the former completely encapsulate quasiparticle braiding properties [20]. Let us suppose that the quasiholes at η_1 and η_2 form our qubit. The quasiholes at η_3 and η_4 will be used to measure and manipulate them. From (4), we see that taking η_3 around η_1 and η_2 results in a factor i in the state $\Psi^{(0)}$, but $-i$ in the state $\Psi^{(1)}$. Taking η_3 around either η_1 or η_2 (but not both) transforms $\Psi^{(0)}$ into $i\Psi^{(1)}$ and vice versa.

It is also possible [26] to verify the logic associated to braiding operations using a few formal properties of the Jones polynomial at $q = \exp(\pi i/4)$. Taking one quasiparticle around the qubit pair ("linking") results in an extra -1 if the qubit is in state $|1\rangle$ (a factor $d = -q - q^{-1}$ also arises regardless of whether or not the quasiparticle encircles the qubit). The Jones polynomial (operator) at $q = \exp(\pi i/4)$ vanishes for the links in Figs. 1(a) and 1(b) by calculation, 1(c) by parity, and is nonvanishing only for 1(d) (which applies to all processes with topologically equivalent link diagrams, e.g., interchanging inputs and outputs so, for example, 1(d) corresponds to four different processes). In case 1(d), the qubit is flipped by the elementary braid operation.

Experimental configuration.—The basic setup which we propose is a quantum Hall bar with two individually-gated antidots in its interior, labeled 1 and 2 in Fig. 2. There are front gates which enable tunneling between A and B at the edges. It is useful to have a third antidot at the point C

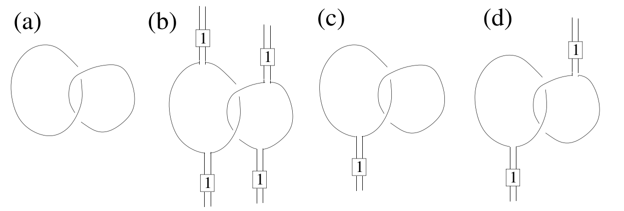

FIG. 1. By evaluating the Jones polynomial at $q = \exp(\pi i/4)$ for these links, we can obtain the desired matrix elements for braiding operations manipulating the qubit. The boxed 1 is a projector on the pair of quasiparticles which puts them in the state $|1\rangle$.

midway between A and B in order to precisely control the charge which tunnels between A and B, but we have not depicted it to avoid clutter. Two more front gates enable tunneling at M and N and at P and Q. There are three basic procedures which we would like to execute: (1) initialize the qubit and measure its initial state, (2) flip the qubit, and (3) measure it again.

In order to initialize the qubit, we first put charge $e/2$ on one of the antidots, say 1. Since the fermionic zero mode is now localized on this antidot, the environment will ''measure'' it, and it will either be occupied or unoccupied (not a superposition of the two). We can determine which state it is in by applying voltage to the front gates at M and N and at P and Q so that tunneling can occur there with amplitudes t_{MN} and t_{PQ}. The longitudinal conductivity, σ_{xx} is determined by the probability for current entering the bottom edge at X in Fig. 2 to exit along the top edge at Y. This is given, to lowest order in t_{MN} and t_{PQ}, by the interference between two processes: one in which a quasiparticle tunnels from M to N; and another in which the quasiparticle instead continues along the bottom edge to P, tunnels to Q, and then moves along the top edge to N. (We subsume into t_{PQ} the phase associated with the extra distance traveled in the second process.) If a neutral fermion is not present, which we will denote by $|0\rangle$, then $\sigma_{xx} \propto |t_{MN} + it_{PQ}|^2$. If it is present, however, which we denote by $|1\rangle$, then $\sigma_{xx} \propto |t_{MN} - it_{PQ}|^2$. Suppose that the initial state of the qubit is $|0\rangle$. Now, let us apply voltage to antidots 1 and 2 so that charge $e/4$ is transferred from 1 to 2. There is now one charge-$e/4$ quasihole on each antidot. The state of the qubit is unaffected by this process.

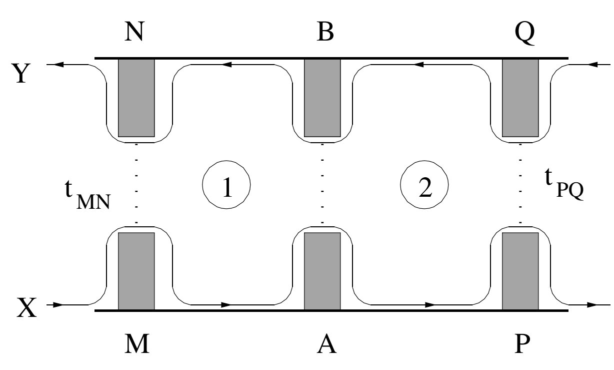

FIG. 2. A schematic depiction of a Hall bar with front gates which enable tunneling between the two edges at M, N and P, Q, thereby allowing a measurement of the qubit formed by the correlation between antidots 1 and 2. Front gates (shaded regions) also allow tunneling at A, B which flips the qubit.

In order to flip this qubit, we now apply voltage to the front gates at A and B so that one charge $e/4$ quasiparticle tunnels between the edges. In order to ensure that only a single quasiparticle tunnels, it is useful to tune the voltage of the antidot at C and the backgate at A so that a single quasiparticle tunnels from the edge to the antidot at C. We can then lower the voltage of the back gate at A so that no further tunneling can occur there and apply voltage to the back gate at B so that the quasiparticle can tunnel from C to B. By this two-step process, we can tunnel a single quasiparticle from A to B. If the $\nu = 5/2$ plateau is in the phase of the Pfaffian state, this will transform $|0\rangle$ to $|1\rangle$. This is our logical NOT operation. The gate which creates the antidot at C must be turned off at the beginning and end of the bit flip process so that there are no quasiparticles there either before or after which could become entangled with our qubit. The topological charge of the boundary of the system (i.e., both edges together) has only two possible values, which must balance that of the qubit. Therefore, extra topological degrees of freedom are not introduced by either the coupling to the leads or interedge tunneling processes.

We can now measure our qubit again by tuning the front gates so that tunneling again occurs between M and N and between P and Q with amplitudes t_{MN} and t_{PQ}. If, as expected, the qubit is now in the state $|1\rangle$ we will find $\sigma_{xx} \propto |t_{MN} - it_{PQ}|^2$. On the other hand, if the $\nu = 5/2$ state were Abelian, σ_{xx} would not be affected by the motion of a quasiparticle from A to B.

In order to execute these steps, it is important that we know that we have one (modulo 4) charge-$e/4$ quasihole on each antidot. This can be ensured by measuring the tunneling conductance G_t^{ad} from one edge to the other through each antidot [17]. As we sweep the magnetic field, there will be a series of peaks in G_t^{ad} corresponding to the passage through the Fermi level of quasihole states of the antidot. The spacing ΔB between states is determined by the condition that an additional state passes through the Fermi level when one additional half-flux-quantum, $\Phi_0/2$ is enclosed in the dot. Thus, the number of quasiholes is given simply by $\lfloor B/\Delta B \rfloor$. Alternatively, with a back gate, we could directly measure capacitatively the charge on each antidot [17].

Estimate of error rate.—Bit flip and phase flip errors, respectively, occur when an uncontrolled charge-$e/4$ quasiparticle performs one of the two basic processes above: encircling one of the antidots (or passing from one edge to the other between them) or encircling both of them. The rate for these processes is related to the longitudinal resistivity (which vanishes within experimental accuracy) because it is limited by the density and mobility of excited quasiparticles. Even without considering the suppression factor associated with the latter (which depends on the ratio of the diffusion or hopping length, a, to the system size, L), we already have a strong upper bound on the error rate following from the thermally-activated form of the former:

$$\frac{\Gamma}{\Delta} \sim \frac{T}{\Delta} e^{-\Delta/T} < 10^{-30}. \tag{5}$$

Here, we have used the best current measured value [27] for the quasiparticle gap $\Delta = 500$ mK of the $5/2$ state and the lowest achieved measurement temperature $T = 5$ mK. For arbitrary braid-based computation, in a more elaborate device, it is sufficient if we further have $e^{\Delta/T} > \nu \Delta L^2$, where ν is the density of states. The effect of residual pinned quasiparticles can be diagnosed and accounted for in software. These error rates are substantially lower than the estimated error rate for any other proposed physical implementations of quantum computation. Compared to other scalable solid state architectures, such as localized electron spin qubits [28] in Si or GaAs nanostructures, where the estimated error rate is around 10^{-4} even in the best possible circumstances, the errors associated with $\nu = 5/2$ quantum Hall anyons is essentially negligible.

The ideal error rate for the $5/2$ state may actually be substantially lower than even this very low currently achievable value of 10^{-30}. There is strong theoretical evidence [29] that the ideal excitation gap ($\sim$2 K) for the $5/2$ quantum Hall state is much larger than the currently achieved gap value of 500 mK. Using an ideal gap of 2 K, we get an astronomically low error rate of 10^{-100}. This expected higher value of Δ ($\sim$2 K) is consistent with the experimental development of the activation gap measurement [13] of the $5/2$ state. The early measurements on fairly modest quality samples (i.e., relatively highly disordered) gave $\Delta \sim 100$ mK whereas recent measurements in extremely high-quality (i.e., low-disorder) samples give $\Delta \sim 300$–500 mK [13]. This implies that the $5/2$ excitation gap is susceptible to strong suppression by disorder as has recently been theoretically argued [29]. Since improvement in sample quality has already led to a factor of 5 enhancement in Δ (from 100 to 500 mK), it is not unreasonable to expect further improvements.

There are, in principle, other sources of error, but we expect them to be of minor significance. For example, if two quasiparticles come close to each other, then their mutual interaction leads to an error (e.g., through the exchange of a virtual particle). Such a virtual exchange is, however, a quantum tunneling process which should be exponentially suppressed if we keep the quasiparticles reasonably far from each other.

We should also mention that recently [13] the $12/5$ fractional quantum Hall state has been observed experimentally in the highest mobility sample at the lowest possible temperatures. This state, thought to be a non-Abelian state related to parafermions [30], is particularly exciting from the perspective of topological quantum computation because its braid group representation is dense in the unitary group [10] making this state an ideal candidate for topological quantum computation. The measured gap value in the $12/5$ state is currently rather small ($\sim$70 mK), making any experimental effort along the line of our discussion in this Letter premature. However, we expect that this is also strongly affected by disorder and that the eventual ideal gap at $12/5$ will be larger.

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