Quasiparticle Approach to the Modulational Instability of Drift Waves Coupling to Zonal Flows

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The interaction between broadband drift mode turbulence and zonal flows has been studied through the wave-kinetic approach. Simulations have been conducted in which a particle-in-cell representation is used for the quasiparticles, while a fluid model is employed for the plasma. The interactions have been studied in a plasma edge configuration which has applications in both tokamak physics and magnetopause boundary layer studies. Simulation results show the development of a zonal flow through the modulational instability of the drift wave distribution, as well as the existence of solitary zonal flow structures about an ion gyroradius wide, drifting towards steeper relative density gradients.

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In this Letter, we describe for the first time numerical simulations using the wave-kinetic description of the modulational instability of drift waves. The numerical model is based on a particle-in-cell (PIC) treatment of quasiparticles, the "drift waves," while we describe the plasma as a fluid. In our work we can easily treat broadband drift mode turbulence, whereas other numerical treatments deal with a monochromatic drift mode pump wave [1,2]. Earlier work on the broadband nature of the turbulence was restricted to linear theory [3]. Our simulations clearly demonstrate the modulation of a broadband spectrum of drift waves into solitary wave packets coupled to lower-frequency electrostatic zonal flows which propagate down the plasma density gradient into a region of lower background density.

In this work, we study broadband drift wave turbulence in a plasma using the wave-kinetic approach [4–8]. In this approach, the turbulent high-frequency waves are treated as a distribution of quasiparticles. Numerically, their propagation can be calculated using, e.g., a particle-in-cell method. The advantages of the wave-kinetic approach are many. It provides a convenient way to introduce a spectrum of turbulent waves into the plasma, as opposed to classical PIC or Vlasov codes which usually only treat monochromatic waves. A wide range of applications can be studied: laser-plasma interactions (laser wakefield studies) [8], drift modes interacting with zonal flows as seen in both tokamak plasma and at the magnetopause, turbulence in planetary and stellar atmospheres, strong plasma turbulence, e.g., Langmuir wave collapse, and so on. Another major advantage is in the range of powerful new diagnostics of the high-frequency waves that come with the wave-kinetic approach. The high-frequency wave modes can be followed individually or collectively, revealing a deep insight in their behavior, individual and collective propagation, and energy, momentum exchange with the background plasma.

The wave-kinetic approach is most suitable for the study of modulational instabilities involving a high-frequency pump wave and a low-frequency plasma wave. However, as shown by Tsytovich [9] or by Smolyakov and Diamond [10], the wave-kinetic approach can also be used for the study of multiwave parametric instabilities. This can be performed by deliberately seeding the Wigner function (8) with low-amplitude discrete sideband waves, not unlike the seeding used in the classical approach to this problem [1,2,11]. In this Letter, however, we will concentrate on the role of the modulational instability. The wave-kinetic approach to parametric instabilities will be the subject of future work.

We consider the modulational instability of drift modes interacting with zonal flows in slab geometry. This problem is highly topical at the moment and is associated with transport barriers in the H-mode confinement scheme for tokamaks. It can also explain the origin of solitary wave structures that are observed at the magnetopause [12,13]. The drift wave problem has been tackled by a number of authors. The work of Smolyakov *et al.* [3] considered the linear theory of broadband drift modes coupled to zonal flows, while Lashmore-Davies *et al.* [1,2] considered a monochromatic treatment with a drift mode pump wave coupling through the zonal flow to a series of discrete Stokes and anti-Stokes sidebands. There is not only a conceptual difference in the treatment of the problem: the two approaches solve different aspects of the nonlinear coupling problem.

In this Letter, we nonlinearly solve the equations described by Smolyakov *et al.* [3] for a broadband distribution of drift modes. In their work, the equations are linearized and a δ distribution (in momentum space) of drift modes is assumed. In this Letter, we make no such assumption and solve for an arbitrary momentum distribution such as a Gaussian.

At the heart of the wave-kinetic model lies the wave mode density function $N(t, x, k)$. This function is defined as $N(t, x, k) \equiv W(t, x, k)/\omega(k)$, where $W(t, x, k)$ denotes the energy density of the high-frequency waves as a function of time, position, and wave vector (wave momentum), while $\omega(k)$ denotes the frequency (energy) of an individual wave mode as given by the linear dispersion relation for the high-frequency waves. A monochromatic or coherent wave is then described by a compact distribution function N with a small spread in $\mathbf{k}$ space, whereas broadband turbulence is described by a widely spread-out distribution function.

In configurations where creation and annihilation of wave modes need not be taken into account, the function N obeys a Vlasov-like equation [Eq. (9) below]. The quantity $\omega(\mathbf{k})$ is again given by the dispersion relation for the high-frequency waves. The action of the quasiparticles on the plasma is usually given by the associated *ponderomotive force* [14] or *Reynolds stress* [3]. We have opted to solve this equation through the particle-in-cell method. For the plasma, one can then employ either a cold fluid model or more advanced models, which take the ponderomotive force as input.

We consider drift waves in a sheared magnetic field $\mathbf{B} = B\mathbf{z}$, such as in a tokamak or at the magnetopause, and use a simple two-dimensional model to describe electron drift waves in the (r, θ) plane. For the background density, we use a 2D Gaussian profile around the origin. Following the approach of Smolyakov *et al.* [3], we write the full electrostatic potential in the form $\phi = \bar{\phi} + \tilde{\phi}$, where $\bar{\phi}$ is the θ-averaged part (having wave number $k_\theta = 0$) and $\tilde{\phi}$ represents the "fluctuation" (having wave number $k_\theta \neq 0$ by definition). With these definitions, $\mathbf{V}_0 = \mathbf{e}_z \times \nabla\bar{\phi}/B$ is the θ-averaged $\mathbf{E} \times \mathbf{B}$ drift (also known as "zonal flow"), while $\tilde{\mathbf{V}}_E = \mathbf{e}_z \times \nabla\tilde{\phi}/B$ represents the fluctuating $\mathbf{E} \times \mathbf{B}$ drift and $\mathbf{V}_d$ is the diamagnetic drift. The nonlinear behavior of the electrostatic potential is then given by the following well-known equation [15–17]:

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla + \mathbf{V}_d \cdot \nabla\right)\frac{e\tilde{\phi}}{T_e} - \rho_s^2\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla + \tilde{\mathbf{V}}_E \cdot \nabla\right)\nabla_\perp^2 \frac{e\phi}{T_e} = 0, \tag{1}$$

where $\rho_s = c_s/\Omega_{ci}$, $c_s = \sqrt{T_e/m_i}$ is the ion sound speed, and $\Omega_{ci} = eB/m_i$ is the ion gyrofrequency. Furthermore, we take the diamagnetic drift velocity $\mathbf{V}_d$ to be given by $\mathbf{V}_d = \nabla p \times \mathbf{B}/(neB^2)$, where n denotes the background plasma density and $p = nT_e$ the electron pressure. Assuming T_e to be constant, and n to depend on r only, we get $\mathbf{V}_d = V_d\mathbf{e}_\theta$ and $V_d = -T_e/(neB)\partial n/\partial r$.

For the remainder of this Letter, we shall apply the following scalings: $t \to \Omega_{ci}t$, $\mathbf{r} \to \mathbf{r}/\rho_s$, $\mathbf{V} \to \mathbf{V}/c_s$, $\phi \to e\phi/T_e$. After scaling, we find that there is only one parameter left in the equations: $V_d = \rho_s/L_n$, where L_n denotes the typical scale length of the background density gradient, given by $L_n^{-1} = -(1/n)\partial n/\partial r$, which may of course depend on r.

Averaging (1) in the θ direction, we obtain

$$\frac{\partial}{\partial t}\nabla_\perp^2\bar{\phi} + \mathbf{V}_0 \cdot \nabla\nabla_\perp^2\bar{\phi} + \overline{(\tilde{\mathbf{V}}_E \cdot \nabla\nabla_\perp^2\tilde{\phi})} = 0. \tag{2}$$

Since $\nabla\bar{\phi} = (\partial\bar{\phi}/\partial r)\mathbf{r}$, we find that the second term in this equation vanishes. Using the Poisson bracket $\{a, b\} \equiv (1/r)(\partial_r a \cdot \partial_\theta b - \partial_\theta a \cdot \partial_r b)$, we rewrite the above equation as

$$\frac{\partial}{\partial t}\nabla_\perp^2\bar{\phi} + \overline{\{\tilde{\phi}, \nabla_\perp^2\tilde{\phi}\}} = 0. \tag{3}$$

Upon subtracting (3) from (1), we find for the evolution of the fluctuations

$$\frac{\partial\hat{f}}{\partial t} + \mathbf{V}_0 \cdot \nabla\hat{f} + \mathbf{V}_d \cdot \nabla\tilde{\phi} = -\frac{1}{r}\frac{\partial\tilde{\phi}}{\partial\theta}\frac{\partial\bar{\phi}_{rr}}{\partial r} + \{\tilde{\phi}, \nabla_\perp^2\tilde{\phi}\} - \overline{\{\tilde{\phi}, \nabla_\perp^2\tilde{\phi}\}}, \tag{4}$$

where $\hat{f} = (1 - \rho_s^2\nabla_\perp^2)\tilde{\phi}$, or $\hat{f} = (1 - \nabla_\perp^2)\tilde{\phi}$ after scaling. For small perturbations $\tilde{\phi} \sim \exp i(\mathbf{k} \cdot \mathbf{x} - \omega_k t)$, we can neglect the nonlinear terms at the right-hand side of (4) to obtain the linear dispersion relation

$$\omega_k = k_\theta V_0 + \omega_k^l = k_\theta V_0 + \frac{k_\theta V_d}{1 + k_\perp^2}, \tag{5}$$

where $\mathbf{k} \to \rho_s\mathbf{k}$ and $\omega_k \to \omega_k/\Omega_{ci}$. For a zonal flow $\bar{\phi} \sim \exp i(\mathbf{q} \cdot \mathbf{x} - \omega t)$, Eq. (2) quickly yields the dispersion relation $\mathrm{Re}(\omega) = 0$.

Since $\{\tilde{\phi}, \tilde{\phi}\} = 0$ and $\tilde{\phi}$ is periodic in θ, it follows that

$$\begin{aligned}\overline{\{\tilde{\phi}, \nabla_\perp^2\tilde{\phi}\}} &= \overline{(1/r)\partial_\theta(\partial_r\tilde{\phi} \cdot \nabla_\perp^2\tilde{\phi})} \\ &\quad - \overline{\partial_r[(1/r)\partial_\theta\tilde{\phi} \cdot \nabla_\perp^2\tilde{\phi}]} - \overline{(1/r^2)\partial_\theta\tilde{\phi} \cdot \nabla_\perp^2\tilde{\phi}}, \\ &= -(\partial_r + 1/r)\overline{[(1/r)\partial_\theta\tilde{\phi} \cdot \nabla_\perp^2\tilde{\phi}]}, \\ &= -(\partial_r + 1/r)(\partial_r + 2/r)\overline{[\partial_r\tilde{\phi} \cdot (1/r)\partial_\theta\tilde{\phi}]}.\end{aligned}$$

This result differs from the one obtained by Smolyakov *et al.* [3], as they left out the $2/r$ term without an explanation. Therefore, their result is expected to be valid only for r much larger than twice the zonal flow wave length. Equation (3) then reduces to

$$\frac{\partial}{\partial r}\frac{\partial\phi}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\overline{\left(\frac{\partial\tilde{\phi}}{\partial r} \cdot \frac{1}{r}\frac{\partial\tilde{\phi}}{\partial\theta}\right)}. \tag{6}$$

As derived in Refs. [10,18], the drift wave/zonal flow system described by (1) has the adiabatic action invariant N_k:

$$N_k = \mathcal{E}_k/\omega_k^l = (1 + k_\perp^2)^2 I_k, \tag{7}$$

$$I_k = \int \tilde{\phi}(t, \mathbf{x} + \mathbf{s}/2)\tilde{\phi}^*(t, \mathbf{x} - \mathbf{s}/2)\exp(-i\mathbf{k} \cdot \mathbf{s})d^2s. \tag{8}$$

Here $\mathcal{E}$ denotes the energy density of the plasma fluid and I_k is the so-called *Wigner function* of the high-frequency drift wave potential $\tilde{\phi}$. Equation (4) can then be rewritten as a Liouville equation for N_k:

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega_k}{\partial \mathbf{k}}\frac{\partial N_k}{\partial \mathbf{x}} - \frac{\partial \omega_k}{\partial \mathbf{x}}\frac{\partial N_k}{\partial \mathbf{k}} = 0. \quad (9)$$

Using the definition of N_k, we can now rewrite (6) as

$$\frac{\partial}{\partial r}\frac{\partial \bar{\phi}}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\int \frac{k_r k_\theta}{(1 + k_\perp^2)^2} N_k d^2k. \quad (10)$$

In Cartesian coordinates, the Hamiltonian equations for the quasiparticles read $d\mathbf{x}/dt = \nabla_k \omega_k$, $d\mathbf{k}/dt = -\nabla_x \omega_k$. As these are tensor equations, their validity does not depend on the choice of coordinates. Thus, they can be directly converted to polar coordinates to yield

$$\frac{dr}{dt} = \frac{\partial \omega_k}{\partial k_r}, \quad \frac{d\theta}{dt} = \frac{1}{r}\frac{\partial \omega_k}{\partial k_\theta}, \quad \frac{dk_r}{dt} = -\frac{\partial \omega_k}{\partial r} + \frac{k_\theta}{r}\frac{\partial \omega_k}{\partial k_\theta},$$
$$\frac{dk_\theta}{dt} = -\frac{1}{r}\frac{\partial \omega_k}{\partial \theta} - \frac{k_r}{r}\frac{\partial \omega_k}{\partial k_\theta}. \quad (11)$$

Using a particle-in-cell model, we approximate N_k by a distribution of macroparticles in phase space, each one representing a certain number of drift waves (cf. photons, or phonons in solid state physics). The equations of motion for each macroparticle are given by $dx/dt = \partial\omega_k/\partial k$ and $dk/dt = -\partial\omega_k/\partial x$, where ω_k is given by (5). The zonal flow acts on the drift modes through ω_k, and the action of the drift modes on the zonal flow is given by (10), where the integral is to be replaced by a sum over all macroparticles. This closes the system of equations.

In our simulations, we have studied the case of a broadband distribution of drift waves in a plasma with a 2D Gaussian density profile centered around $r = 0$, as one might find in a tokamak. Note that, with only nonessential modifications, this configuration can also be used to model drift waves at a plasma boundary such as Earth's magnetopause. The initial drift mode distribution is homogeneous in (r, θ) space and Gaussian in (k_r, k_θ) space, with a mean k value of $3/\rho_s$ and a spread of $1/\rho_s$. The following phenomena are observed. (See also Fig. 1.) First, a zonal flow is excited through the modulational instability. Since the zonal flow has zero frequency and thus zero phase velocity, it can only be driven resonantly by the drift waves in regions where the drift mode group velocity is small, i.e., where $V_d = -(1/n)(\partial n/\partial r)$ is small. No zonal flow growth is expected in regions of large background density gradients, i.e., large V_d. This is confirmed by the simulations: there is a large zonal flow growth for small r, where $-(1/n)(\partial n/\partial r)$ is small, and little or no growth for large r, where $-(1/n)(\partial n/\partial r)$ is large. Second, in the plots of the density perturbation δn, we observe solitary wave structures having a width of about $1.5\rho_s$, which break away

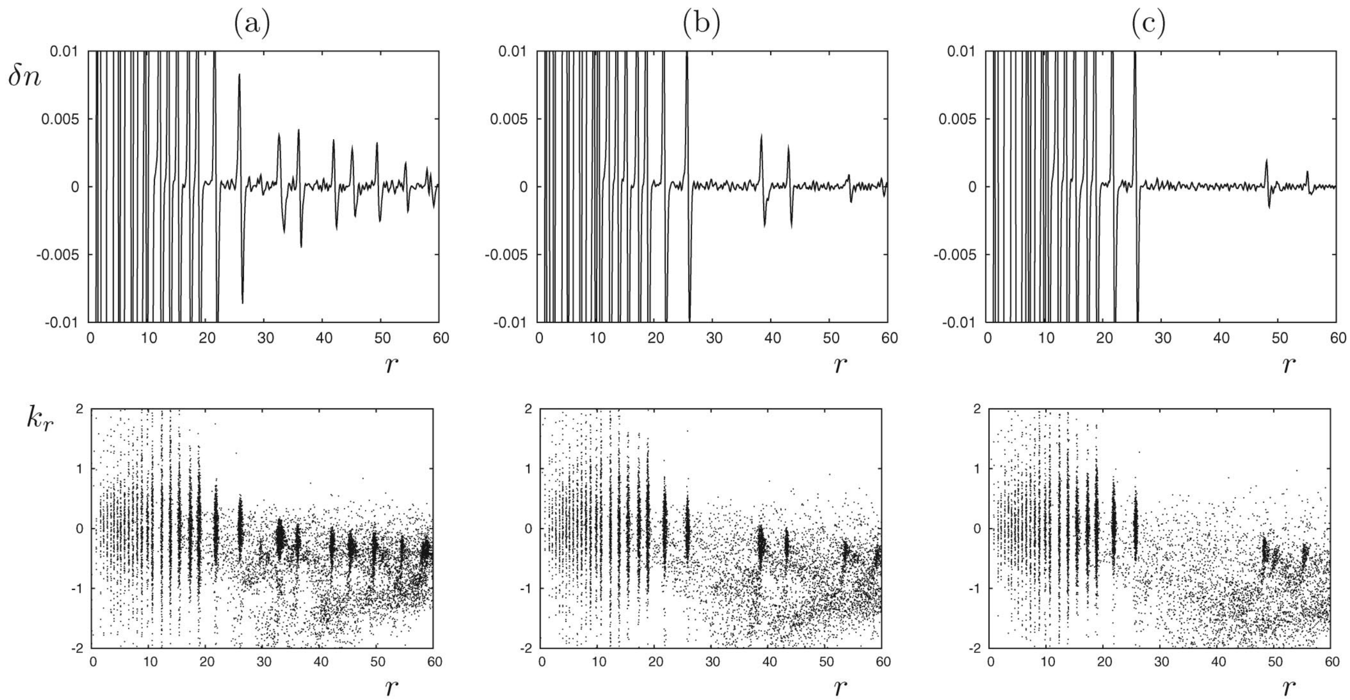

FIG. 1. Simulation results for drift modes interacting with a plasma at $\Omega_{ci}t = 12\,000$ (a), 16 000 (b), and 20 000 (c). The background density gradient is shallow for small r and steep for large r. The top row shows the plasma density perturbations, while the drift mode phase space can be seen in the bottom row. The excitation of a zonal flow through the modulational instability occurs for only small r, where the density gradient is shallow. Solitary wave structures about $1.5\rho_s$ wide that are breaking off the main zonal flow and drifting down the density gradient can be observed.

from the main zonal flow and propagate into regions where no zonal flow developed earlier because the density gradient is too steep and the drift modes are out of resonance with the zonal flow. In quasiparticle phase space, clumping of drift modes occurs as a result of the modulational instability. Note that quasiparticles from the same clump stay together rather long, while the clumps may propagate independently of each other. Obviously, the solitary structures seen in the plots of δn are caused by individual clumps of drift modes moving away from the main zonal flow into a region without zonal flow.

In Fig. 1, the density perturbation δn of the background plasma and the radial wave number k_r of the drift modes are plotted versus radius r. For a Gaussian density profile, we have $L_n \sim 1/r$ and $V_d \sim r$, which explains why the zonal flow growth is largest for low r and why it propagates towards high r. Solitonlike structures in δn, driven by clumps of drift modes, drift towards large r independently of each other. The clumps of drift modes manage to retain their identity for quite a long time.

The formation of these solitary structures can be explained as follows. The two main contributions to the velocity of the drift modes are the $\mathbf{E} \times \mathbf{B}$ drift $V_0 \sim \partial \bar{\phi}/\partial r$ and the diamagnetic drift $V_d \sim -(1/n_0)\partial n_0/\partial r$. The $\mathbf{E} \times \mathbf{B}$ drift follows the perturbation in $\bar{\phi}$ and causes the solitary structure to stay together. The diamagnetic drift acts in the same direction for all drift modes with the same sign of k_θ, and increases with the relative density gradient. This drift may cause drift modes on a density slope to move apart, thus opposing the effect of the $\mathbf{E} \times \mathbf{B}$ drift. Thus, a gradient in V_d will exert a tidal force on neighboring clumps of drift waves. This force, even if it is not large enough to pull individual clumps apart, can cause a clump of drift modes that originated at a slightly steeper density gradient to move away from the others and towards regions with even steeper density gradients. This explains the solitary structures moving into zonal flow free regions, as seen in the simulations. As long as the tidal force is relatively small across one such structure, the structure will stay together and propagate as a whole.

The theoretical and numerical results presented in this Letter may have important consequences for tokamak physics. In high-confinement modes in tokamaks, the formation of turbulence on the boundary layer between the high-density and low-density regions of the plasma has been observed, as well as solitary structures, so-called "plasma blobs" [19] or edge-localized modes [20], moving towards the low-density region. The above exposé might very well serve to explain this behavior. The occurrence of similar phenomena at plasma boundary layers in Earth's magnetosphere, such as the magnetopause, has also been observed [12] and is explained by our model [13].

We have studied the modulational instability of a broadband, incoherent distribution of drift waves and a coherent zonal flow in a magnetized plasma through the wave-kinetic approach. Simulations have been performed for configurations mimicking conditions in a tokamak plasma. In the simulations, excitation of a zonal flow through the modulational instability of the drift waves can be seen, as well as the formation of solitary wave structures separating themselves from the main zonal flow.

The possible consequences of these results are manifold, as plasma turbulence and solitary wave structures at density transitions can be observed in configurations ranging from high-confinement tokamak plasmas to boundary layers in Earth's magnetosphere. Using the wave-kinetic approach to drift mode-zonal flow interaction, the origin and the behavior of these structures are now understood to be the result of the nonlinear evolution of the drift mode modulational instability. As a consequence, a wide range of applications for the wave-kinetic treatment of drift modes is anticipated. This result therefore clearly demonstrates the power of the wave-kinetic approach in the study of broadband turbulence.

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