Construction of Exact Plasma Equilibrium Solutions with Different Geometries

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Infinite families of exact isotropic and anisotropic plasma equilibria with and without dynamics can be constructed in different geometries, using the representation of the static MHD equilibrium system in coordinates connected with magnetic surfaces. A sample equilibrium anisotropic model of Earth magnetosheath plasma is given.

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The isotropic magnetohydrodynamics (MHD) equations [1] and Chew-Golberger-Low (CGL) anisotropic plasma equations [2] are widely used in thermonuclear fusion, astrophysics, and geophysics research [1–4].

For many applications, equilibrium and quasiequilibrium plasma configurations are of particular interest. The search for exact equilibrium solutions to MHD and CGL systems has been going on during recent decades. Several types and families were found, primarily in the MHD framework, using mainly reductions by symmetry groups (as Grad-Shafranov and JFKO equations for axial and helical symmetry) and Euler potentials (see [3,5–8]). A CGL analog of the Grad-Shafranov equation is known [9] but has not been widely used due to its complexity.

Recently, infinite groups of symmetries of MHD and CGL equilibrium systems [10–13] and an infinite-parameter set of transformations from MHD to CGL equilibria [12] have been found. These symmetries and transformations alter the physical parameters of plasma equilibria but preserve the solution topology. For effective analytical modeling of physical phenomena, a greater diversity of exact solutions with different geometries is required, but only a limited number is available.

We report a rather general method of construction of exact plasma equilibria having various types of magnetic field geometry and different physical properties. Combined with available infinite symmetries and transformations, new solutions provide a tool for modeling a variety of plasma phenomena with exact solutions in MHD and CGL frameworks. The method also allows the construction of exact time-independent Euler and irrotational Navier-Stokes fluid flows. The approach illustrates the importance of knowledge and explicit use of the topological properties in the analysis of physical partial differential equation systems with several spatial variables.

The incompressible MHD equilibrium equations are

$$\rho \mathbf{V} \times \operatorname{curl} \mathbf{V} - \mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad} P + \rho \operatorname{grad} \mathbf{V}^2/2 \quad (1)$$

$$\operatorname{div} \rho\mathbf{V} = 0, \quad \operatorname{curl}(\mathbf{V}\times\mathbf{B}) = 0, \quad \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{V} = 0. \quad (2)$$

In strongly magnetized or rarified plasmas, the pressure P is replaced by a 3×3 tensor, giving rise to the CGL plasma equilibrium model [2,12]:

$$\begin{aligned}\rho\mathbf{V}\times\operatorname{curl}\mathbf{V} - (1-\tau)\mathbf{B}\times\operatorname{curl}\mathbf{B} = \operatorname{grad} p_\perp + \rho\operatorname{grad}\mathbf{V}^2/2 \\ + \tau\operatorname{grad}\mathbf{B}^2/2 \\ + \mathbf{B}(\mathbf{B}\cdot\operatorname{grad}\tau), \quad (3)\end{aligned}$$

$$\operatorname{div} \rho\mathbf{V} = 0, \qquad \operatorname{curl}(\mathbf{V}\times\mathbf{B}) = 0, \qquad \operatorname{div}\mathbf{B} = 0; \quad (4)$$

$\tau = (p_\parallel - p_\perp)/\mathbf{B}^2$ is the anisotropy factor; $p_\parallel$ and $p_\perp$ are pressure components along the magnetic field and in the transverse direction. The MHD system (1) and (2) requires one equation of state; the CGL system needs two. An incompressibility condition $\operatorname{div}\mathbf{V} = 0$ is often imposed as one of such equations.

A general MHD or CGL equilibrium possesses a family of nested 2D magnetic surfaces, to which velocity $\mathbf{V}$ and magnetic field $\mathbf{B}$ are tangent. It is also the case for all static ($\mathbf{V} = 0$) MHD equilibrium reductions except Beltrami-type ones [10,12,14]. In many cases, an *orthogonal* coordinate system can be constructed, with one of the coordinates constant on magnetic surfaces [15]. In such coordinates, the static plasma equilibrium system

$$\mathbf{J}\times\mathbf{B} = \operatorname{grad} P, \qquad \operatorname{div}\mathbf{B} = 0, \qquad \operatorname{curl}\mathbf{B} = \mathbf{J}, \quad (5)$$

(which is equivalent to the equilibrium Euler equations of incompressible fluid motion) takes a particular form, and exact solutions are often found. If (u_i), $i = 1, 2, 3$ are orthogonal coordinates, and u_3 enumerates magnetic surfaces, then the pressure is $P(\mathbf{r}) = P(u_3)$, and the magnetic field $\mathbf{B} = B^i\mathbf{e}_i$, $i = 1, 2$. (From here on, summation in the repeated indices from 1 to 2 is assumed.) Using the form of linear differential operators in orthogonal coordinates [16], we find from (5) that there exists a potential $\Phi = \Phi(u_1, u_2, u_3)$:

$$\mathbf{B} = \frac{1}{\sqrt{g_{ii}}}\frac{\partial\Phi}{\partial u_i}\mathbf{e}_i \equiv \operatorname{grad}_{(u_1,u_2)}\Phi, \quad (6)$$

where (u_1, u_2) means that only (u_1, u_2) parts of the operator are used. Here $g_{ij} = g_{ii}^2\delta_{ij}$, $i, j = 1, 2, 3$ is the metric tensor. The electric current density is

$$\begin{aligned}\mathbf{J} &= \operatorname{curl}\mathbf{B} \\ &= -\frac{1}{\sqrt{g_{22}g_{33}}}\frac{\partial^2\Phi}{\partial u_2\partial u_3}\mathbf{e}_1 + \frac{1}{\sqrt{g_{11}g_{33}}}\frac{\partial^2\Phi}{\partial u_1\partial u_3}\mathbf{e}_2. \quad (7)\end{aligned}$$

In coordinates (u_i), u_1-, and u_2- projections of the first (vector) equation of (5) vanish identically; the u_3- projection and the equation div $\mathbf{B}=0$ become

$$\frac{1}{g_{ii}}\frac{\partial\Phi}{\partial u_i}\frac{\partial^2\Phi}{\partial u_i\partial u_3}=-P'(u_3); \quad (8)$$

$$\frac{\partial}{\partial u_i}\left(\frac{g}{g_{ii}}\frac{\partial\Phi}{\partial u_i}\right)=0, \qquad g=\sqrt{g_{11}g_{22}g_{33}}. \quad (9)$$

Hence the system of four equations (5) rewrites as two equations for two unknown functions $\Phi(u_1,u_2,u_3)$, $P(u_3)$. The equation (8) may be written as

$$\text{grad}_{(u_1,u_2)}\Phi\cdot\text{grad}_{(u_1,u_2)}(\partial\Phi/\partial u_3)=-P'(u_3),$$

and (9) is a (u_1,u_2) part of the Laplace equation $\Delta\Phi=0$.

In coordinates with $g_{11}=g_{11}(u_1,u_2)$, $g_{22}=g_{22}(u_1,u_2)$, (8) has a direct interpretation. It simplifies to $\partial/\partial u_3\times(\mathbf{B}^2/2+P)=0$, meaning that the gradient of total plasma energy in the direction transverse to magnetic surfaces vanishes. The total energy of such configurations is finite if and only if the plasma domain is bounded in the direction transverse to magnetic surfaces. For example, for cylindrically symmetric Grad-Shafranov equilibria, with domains unbounded in cylindrical radius r and zero polar component of the magnetic field, in every layer $c_1<z<c_2$, the total energy is infinite (though the magnetic energy may be finite). The same is true for the solutions obtained in, e.g., [3].

We now list several classes of orthogonal coordinates in which explicit solutions of equations (8) and (9) are readily found. We include into consideration force-free [curl $\mathbf{B}=\alpha(\mathbf{r})\mathbf{B}$] and even "vacuum" magnetic fields (curl $\mathbf{B}=0$, $P=$ const). These fields are of interest for physical modeling because they can serve as *starting solutions* in infinite transformations (15) and (16) to produce families of physically nontrivial MHD and CGL equilibria.

Case 1.—$\Phi(u_1,u_2,u_3)$ essentially depends on the magnetic surface coordinate u_3. [Thus the current density $\mathbf{J}$ (7) is nonzero.] Then equations (8) and (9) have a solution

$$\Phi_1(u_1,u_2,u_3)=C_1(u_3)\nu(u_1)+C_2(u_3)\lambda(u_2), \quad (10)$$

when the metric tensor components satisfy [17]

$$g_{11}/g_{22}=\Pi_{i=1}^{3}a_i^2(u_i),$$
$$g_{33}=\mathcal{F}^2[u_3,\lambda(u_2)-\nu(u_1)C_2(u_3)a_3^2(u_3)/C_1(u_3)], \quad (11)$$

with $[C_1^2(u_3)]'+a_3^2(u_3)[C_2^2(u_3)]'=0$, $\nu'(u_1)=a_1(u_1)$, $\lambda'(u_2)=1/a_2(u_2)$. The expression (10) defines a force-free equilibrium curl $\mathbf{B}=\alpha(\mathbf{r})\mathbf{B}$, $P=$ const, with $\alpha(\mathbf{r})=\alpha(u_3)=C_1'(u_3)/\mathcal{F}C_2(u_3)$. Here $a_i(u_i)$, $\mathcal{F}$, $C_k(u_3)$ are arbitrary sufficiently smooth functions.

Moreover, if the metric tensor of coordinates (u_i) satisfies more restricted relations $g_{11}/g_{22}=a_1^2(u_1)a_2^2(u_2)$, $g_{33}=\mathcal{H}^2(u_3)$, then the previous solution is extended: any linear combination of

$$\Phi_2(u_1,u_2,u_3)=\int t(k)e^{\sigma n(k)\nu}\{D_1(u_3)\cos[n(k)\lambda]+D_2(u_3)\sin[n(k)\lambda]\}dk, \quad (12)$$

$$\Phi_3(u_1,u_2,u_3)=\int t(k)e^{\sigma n(k)\lambda}\{D_1(u_3)\cos[n(k)\nu]+D_2(u_3)\sin[n(k)\nu]\}dk, \quad (13)$$

defines a force-free plasma equilibrium with magnetic field (6) and coefficient $\alpha(\mathbf{r})=D_1'(u_3)/\mathcal{H}D_2(u_3)$. Here $d/du_3[D_1^2(u_3)+D_2^2(u_3)]=0$, $\nu(u_1)$ and $\lambda(u_2)$ are same as above; $\sigma=\pm1$; $n(k)$ is an arbitrary function, and $t(k)$ is an arbitrary generalized function. [$n(k)$, $t(k)$ must be chosen so that the integrals converge.]

Case 2.—When $\Phi(u_1,u_2,u_3)$ is independent of u_3, the equation (8) vanishes, provided $P=$ const. The remaining equation (9) is the $(u_1,u_2)-$ part of the 3D Laplace equation. Therefore, in any coordinates where the full 3D Laplace equation $\Delta\phi(u_1,u_2,u_3)=0$ admits a 2D solution $\phi(u_1,u_2)$, there exists a corresponding gradient ("vacuum") magnetic field configuration

$$\text{div }\mathbf{B}=0, \qquad \text{curl }\mathbf{B}=0 \quad (14)$$

with $\mathbf{B}$ tangent to surfaces $u_3=$ const.

Many coordinate systems that admit geometrically nontrivial two-dimensional solutions of the Laplace equation are known [16]. Necessary and sufficient conditions for separability of Laplace equation and the existence of 2D solutions are available [18]. New coordinates where Laplace equation admits 2D solutions can be found using conformal transformations of the complex plane (see [16]).

Any force-free or vacuum plasma equilibrium solution $\{\mathbf{B}_0,P_0\}$ defined by (10), (12), and (13), or (14) can be transformed into physically nontrivial infinite families [10] of dynamic MHD equilibria

$$\mathbf{B}_1=m(\mathbf{r})\mathbf{B}_0,\quad \mathbf{V}_1=n(\mathbf{r})\mathbf{B}_0/(a(\mathbf{r})\sqrt{\rho}),\quad \rho_1=a^2(\mathbf{r})\rho,\quad P_1=CP_0+(C\mathbf{B}_0^2-\mathbf{B}_1^2)/2,\quad m^2(\mathbf{r})-n^2(\mathbf{r})=C=\text{const}, \quad (15)$$

with the same set of magnetic field lines [$a(\mathbf{r})$, $m(\mathbf{r})$, $\rho(\mathbf{r})$ are arbitrary functions constant on magnetic field lines].

In a similar way, the original force-free or vacuum equilibrium, or its dynamic isotropic extension (15), can be further transformed into an infinite family of anisotropic CGL equilibria. Given an MHD equilibrium $\{\mathbf{V},\mathbf{B},P,\rho\}$ with density ρ constant on magnetic field lines and streamlines, formulas

$$\mathbf{B}_2=f(\mathbf{r})\mathbf{B},\quad \mathbf{V}_2=g(\mathbf{r})\mathbf{V},\quad \rho_2=C_0\rho/g^2(\mathbf{r}),\quad p_{\perp2}=C_0P+C_1+[C_0-f^2(\mathbf{r})]\mathbf{B}^2/2,\quad p_{\parallel2}=C_0P+C_1-[C_0-f^2(\mathbf{r})]\mathbf{B}^2/2 \quad (16)$$

define exact CGL plasma equilibria [12]. $f(\mathbf{r})$ and $g(\mathbf{r})$ are arbitrary functions constant on the magnetic field lines and streamlines; $C_0, C_1=$ const.

Transformations (15) and (16) are infinite dimensional and correspond to Abelian Lie groups of symmetries G_m and G_c of MHD and CGL equilibrium equations (1), (2) and (3), (4), respectively; the numbers of connected components of G_m and G_c are 8 and 16 [12]. These groups can be found using the classical Lie group analysis [13].

Sequential application of (15) and (16) produces exact anisotropic CGL equilibria with various relations between pressure tensor components. The resulting solutions, under the proper choice of transformation parameters, are free from firehose and mirror instabilities [12].

A special type of nontrivial dynamic MHD solutions arising from (14) is Chandrasekhar-type MHD equilibria $\mathbf{V} = \pm\mathbf{B}/\sqrt{\rho}$, $P = C_0 - \mathbf{B}^2/2$, which are invariants of Bogoyavlenskij symmetries (15) [11].

Under certain conditions on the metric, equilibrium vacuum magnetic fields (14) can be extended with a Killing component in the u_3- direction.

In particular, if $\Delta\phi(u_1, u_2) = 0$ and $g_{11} = g_{11}(u_1, u_2) = g_{22}$, $g_{33} = g_{33}(u_3)$, then not only the magnetic field (6) with $P(u_3) = \text{const}$ solves the static MHD equations (5), but so does the extended field

$$\mathbf{B}_K^{(1)} = \sum_{i=1}^{2} \frac{1}{\sqrt{g_{ii}}} \frac{\partial\phi}{\partial u_i} \mathbf{e}_i + K(u_1, u_2)\mathbf{e}_3 \qquad (17)$$

with pressure $P = C - K^2(u_1, u_2)/2$. This equilibrium is neither vacuum nor force free. The function $K(u_1, u_2)$ is a harmonic conjugate of $\phi(u_1, u_2)$: $\Delta K(u_1, u_2) = 0$, $\text{grad}\,\phi(u_1, u_2)\cdot\text{grad}\,K(u_1, u_2) = 0$.

Also, when $g_{11} = g_{11}(u_1, u_2)$, $g_{22} = g_{22}(u_1, u_2)$, $g_{33} = a^2(u_3)\mathcal{M}^2(u_1, u_2)$, not only the magnetic field (6) solves the static MHD equations (5), but so does

$$\mathbf{B}_K^{(2)} = \sum_{i=1}^{2} \frac{1}{\sqrt{g_{ii}}} \frac{\partial\phi}{\partial u_i} \mathbf{e}_i + \frac{D}{\sqrt{g_{33}}}\mathbf{e}_3; \qquad D = \text{const.} \quad (18)$$

Here, $\text{div}\,\mathbf{B}_K^{(2)} = 0$, $\text{curl}\,\mathbf{B}_K^{(2)} = 0$, $P = \text{const}$.

Example 1. Nonsymmetric cylindrical plasma equilibria with current sheets.—In Cartesian coordinates (x, y, z), we use a conformal transformation of the plane (x, y): $Z' = a\cosh Z$ $(x = a\cosh u_1 \cos u_2,\ y = a\sinh u_1 \sin u_2,\ z = u_3)$. For the coordinates (u_i), $g_{11} = g_{22} = (\partial x/\partial u_1)^2 + (\partial y/\partial u_1)^2$, $g_{33} = 1$. This metric satisfies necessary conditions for solutions (12) and (13) to exist. We pick a solution (12) with $n(k) = k, t(k) = 1/(0.1k + 1), D_2(u_3) = 0$ extended with a linear term: $\Phi(u_1, u_2) = \int_1^2 t(k) \times (\sinh(ku_1)\cos(ku_2) - 3u_2)dk$. Its harmonic conjugate is $K(u_1, u_2) = \int_1^2 t(k)(\cosh(ku_1)\sin(ku_2) + 3u_1)dk$. The magnetic field (6) is parallel to the $(u_1, u_2)-$ plane and tangent to levels of $K(u_1, u_2) = \text{const}$.

When extended with a Killing component in u_3- direction, it gives rise to a plasma equilibrium with magnetic field $\mathbf{B}_K$ (17) and $\mathbf{J}_K = \text{curl}\,\mathbf{B}_K$, $P_K = C - K^2(u_1, u_2)/2$. This magnetic field is tangent to a family of cylinders of nonsymmetric section $K(u_1, u_2) = \text{const}$ along z axis; it is nonplanar and has nonzero pressure and current density. Cross sections of several magnetic surfaces are shown on Fig. 1. The configuration is well defined in the region between any two cylinders (e.g., $K = 0.8$ and $K = 1.15$). The plasma domain can be restricted to this region, with $\mathbf{B} = 0$ outside, by introducing a boundary surface current $\mathbf{i}_b(\mathbf{r}_1) = \mathbf{B}(\mathbf{r}) \times \mathbf{n}_{\text{out}}(\mathbf{r}_1)$, where $\mathbf{r}_1$ is a point on the boundary of the plasma domain, and $\mathbf{n}_{\text{out}}$ is an outward normal.

Subsequent application of symmetries (15) and/or MHD → CGL transformations (16) maps this equilibrium into families of, respectively, MHD and CGL plasma equilibria, with the same magnetic field lines, but different values of physical parameters [magnetic field, velocity, current, density, and pressure(s)]. These parameters are controlled by the choice of values of arbitrary functions on each magnetic field line. These solutions model hollow cylindrical jetlike plasma structures bounded by current sheets and stretched in z direction.

Example 2. Model of Earth magnetosheath plasma.—In this example, we construct a family of generally nonsymmetric plasma equilibria modeling the equilibrium solar wind flow in the Earth magnetosheath near the stagnation point, past the bow shock. In prolate spheroidal system of coordinates $(u_1, u_2, u_3) = (\theta, \phi, \eta)$, surfaces $\eta = \text{const}$ define prolate spheroids [16]. The 3D Laplace equation in this system is separable and admits solutions $\Phi = H(\eta)T(\theta)$, $H(\eta) = A_1\mathcal{P}_p(\cosh\eta) + B_1\mathcal{L}_p(\cosh\eta)$, $T(\theta) = A_2\mathcal{P}_p(\cos\theta) + B_2\mathcal{L}_p(\cos\theta)$, where $\mathcal{P}_p(z)$, $\mathcal{L}_p(z)$ are Legendre wave functions of first and second kind.

The necessary boundary conditions for the magnetic field $\mathbf{B}$ (6) of the solar wind are: (i) $\lim_{|\mathbf{r}|\to\infty}\mathbf{B} = M_0\mathbf{e}_z$, (ii) $\mathbf{B}\cdot\mathbf{e}_\eta|_{\eta=\eta_0} = 0$. A particular solution satisfying these conditions is $\Phi_0 = [A_1\cosh\eta + B_1 Q_1(\cosh\eta)]\cos\theta$, where $Q_1(x)$ is a Legendre function of the second kind, and

$$B_1 = (\cosh^2\eta_0 - 1)\ln\frac{\cosh\eta_0 + 1}{\cosh\eta_0 - 1} - 2\cosh\eta_0 = \text{const.}$$

The resulting magnetic field (6) is axially symmetric and well defined in the domain $\eta > \eta_0$. Sample field lines in the vicinity of the magnetopause are shown on Fig. 2.

After the application of Bogoyavlenskij symmetries (15) to the vacuum configuration $P_0 = \text{const}$, $\mathbf{B}_0 = \text{grad}_{(u_1,u_2)} \times \Phi_0$, one gets a family of *isotropic dynamic equilibria*

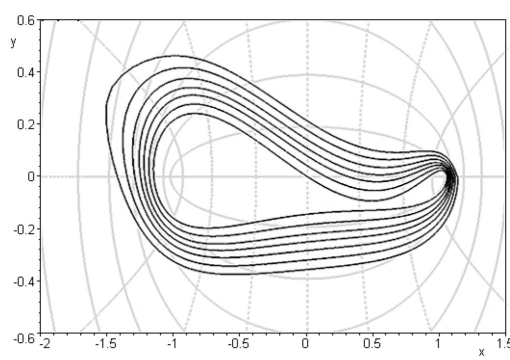

FIG. 1. Noncircular cylindrical magnetic surfaces [levels $K(u_1, u_2) = 0.8, 0.9, 1, 1.1, 1.15$]. Elliptic coordinates (u_1, u_2) shown in grey.

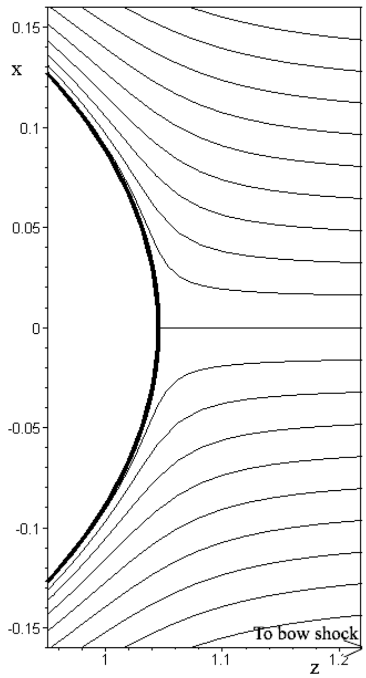

FIG. 2. Magnetic field lines in the Earth magnetosheath model. (Constants: $a = 1$, $M_0 = 1$, $\eta_0 = 0.3$, $A_1 = 1$; a is the coordinate system parameter.)

$$\mathbf{B}_1 = m(\mathbf{r})\mathbf{B}_0, \qquad \mathbf{V}_1 = n(\mathbf{r})\mathbf{B}_0/a(\mathbf{r})\sqrt{\rho_0(\mathbf{r})},$$
$$\rho_1 = a^2(\mathbf{r})\rho_0(\mathbf{r}), \qquad P_1 = CP_0 - n^2(\mathbf{r})\mathbf{B}_0^2/2, \tag{19}$$
$$m^2(\mathbf{r}) - n^2(\mathbf{r}) = C = \text{const.}$$

Here $\mathbf{V}_1 \parallel \mathbf{B}_1$, and $a(\mathbf{r})$, $m(\mathbf{r})$, $\rho_0(\mathbf{r})$ are arbitrary functions constant on magnetic field lines.

To construct an *anisotropic* model, the transformations (16) are used. The resulting CGL equilibrium ($\mathbf{B}_2$, $\mathbf{V}_2$, $p_{\parallel 2}$, $p_{\perp 2}$, ρ_2) is defined by ($P_0 = 0$):

$$\mathbf{B}_2 = f(\mathbf{r})m(\mathbf{r})\mathbf{B}_0, \qquad \mathbf{V}_2 = g(\mathbf{r})n(\mathbf{r})\mathbf{B}_0/a(\mathbf{r})\sqrt{\rho_0(\mathbf{r})},$$
$$\rho_2 = C_0 a^2(\mathbf{r})\rho_0(\mathbf{r})/g^2(\mathbf{r}),$$
$$p_{\perp 2} = C_1 + [C_0 C - f^2(\mathbf{r})m^2(\mathbf{r})]\mathbf{B}_0^2/2, \tag{20}$$
$$p_{\parallel 2} = C_1 + \{f^2(\mathbf{r})m^2(\mathbf{r}) - C_0[C + 2n^2(\mathbf{r})]\}\mathbf{B}_0^2/2.$$

Depending on the choice of values of the arbitrary functions $a(\mathbf{r})$, $m(\mathbf{r})$, $\rho_0(\mathbf{r})$, $f(\mathbf{r})$, $g(\mathbf{r})$ on each magnetic field line, the configuration is axially symmetric or nonsymmetric. This equilibrium is free from firehose instability when $C_0 > 0$, and from mirror instability under the proper choice of the range of $f(\mathbf{r})$ [12].

The general relation between the pressure components of anisotropic pressure tensor is

$$p_{\perp 2}/p_{\parallel 2} = 1 + \kappa\mathbf{B}_2^2/2p_{\parallel 2}, \quad \kappa = C_0/f^2(\mathbf{r}) - 1; \tag{21}$$

its particular form is determined by the choice of $f(\mathbf{r})$, $m(\mathbf{r})$. The relation (21) and the incompressibility condition div $\mathbf{V} = 0$ play the role of two equations of state that close the general CGL equilibrium system. Numerical modeling of anisotropic magnetosheath plasma using the CGL approximation and similar equations of state has been performed in [4].

The suggested approach will be used for the construction of fully three-dimensional static and dynamic plasma equilibrium models in MHD and CGL frameworks. Solutions with and without geometrical symmetries, for plasma domains of different shapes, are available. For anisotropic plasmas, model solutions with different relations between pressure components can be chosen.

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