Connecting Black Holes and Black Strings

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Static vacuum spacetimes with one compact dimension include black holes with localized horizons but also uniform and nonuniform black strings where the horizon wraps over the compact dimension. We present new numerical solutions for these localized black holes in 5 and 6 dimensions. Combined with previous 6D nonuniform string results, these provide evidence that the black hole and nonuniform string branches join at a topology changing solution.

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Introduction and summary.—The task of this Letter is to resolve the structure of static vacuum solutions of Kaluza-Klein theory, namely, pure gravity compactified on a circle. Let us firstly motivate our interests in this problem.

Many scenarios in string theory have extra dimensions large enough that they may be described geometrically [1,2]. In such models it is important to understand the behavior and types of black hole solutions, and, in particular, whether there are potentially new signals from this physics [3].

The problem is also interesting as it is connected by holography to the phase structure of large N_c super Yang-Mills theory, compactified on a circle, at strong 't Hooft coupling [4–9]. Since strong coupling results are scarce for this field theory, gravity provides the only window into this regime. Furthermore, the same phase structure predicted by gravity at strong coupling appears to persist to weak coupling [8].

Lastly, this Letter provides evidence that for Kaluza-Klein theory the 3 branches of solutions—the localized black holes, uniform, and nonuniform strings—are connected in an elegant way, initially conjectured by Kol who postulated the problem is controlled by one relevant order parameter [10]. Learning more about this Morse-theory-inspired approach may shed light on the new and exotic phenomena found in higher dimensions [11–13].

For small masses, localized black holes (BH) should look like Schwarzschild solutions. Increasing their mass they deform as they feel the compact circle [14,15]. The key question is then whether there is a maximum size for these solutions beyond which they no longer "fit."

In d spacetime dimensions, with $d \geq 5$, uniform string (US) solutions exist. These are direct products of $(d-1)$ dimensional Schwarzschild with the circle, and are the only uncharged solutions with a horizon that is asymptotically $R^{1,d-2} \times S^1$ which has a simple analytic form (except for bubble spacetimes which we do not consider here [16]).

The nonuniform strings (NUS) were discovered when Gregory and Laflamme showed that for a given circle size, uniform strings below a critical mass are linearly unstable [17,18]. At the critical point there is a static mode breaking translation invariance on the circle. Motivated by dynamical considerations [19], Gubser showed this mode remains static to all orders in perturbation theory [20].

Kol conjectured [10] that the "waist" of the nonuniform string would shrink to nothing, locally forming a cone geometry, which connected to the black hole branch by resolving the cone to change the horizon topology (see also [21]). Elliptic numerical methods were used to construct the nonuniform strings in 6D [22,23] and the cone geometry was seen to emerge for the most nonuniform solutions [24]. These methods have recently been applied to construct the black hole solutions in 5D and 6D, and thus in principle we can test Kol's conjecture from the "other side" [25,26].

Here we present new numerical solutions for the 5D and 6D localized black holes which significantly improve on the previous works [25,26] (see footnote [27]). In both dimensions they behave similarly, and we find a maximum size localized black hole that can "fit" in the circle dimension. The 6D results provide evidence that the nonuniform and black hole branches do indeed merge at a topology changing solution.

Method.—Both the nonuniform strings and black holes are static axisymmetric geometries that can be written in the form,

$$ds^2 = -e^{2A}dt^2 + e^{2B}(dr^2 + dz^2) + e^{2C}r^2 d\Omega_{d-3}^2, \quad (1)$$

where A, B, C are functions of r, z. We take these to vanish at large radial coordinate r, and hence the geometry is asymptotically $R^{1,d-2} \times S^1$. The circle coordinate z has length L asymptotically. We then employ a numerical method developed in [22,28,29] which uses relaxation techniques to solve for A, B, C while ensuring all the Einstein equations are satisfied. The reader is referred to [22] for details of the procedure [30].

Following [32–35], because the solutions are not asymptotically flat, they are characterized in terms of 2 asymptotic charges: the mass M and a dimensionless tension n, which then give a first law, $dM = TdS + nMdL/L$ where T, S are the black hole temperature and entropy.

In the following plots, we will fix the asymptotic circle length $L = 1$ for all solutions shown, and then use the quantity n to characterize the solutions. For small black holes $n \simeq 0$, and for uniform strings $n = 1/(d-3)$. For convenience we normalize the thermodynamic quantities T, S, M, n by their value for the critical uniform string, $T_{\rm crit}$, $S_{\rm crit}$, $M_{\rm crit}$, $n_{\rm crit}$.

The most error prone part of the numerical calculation is extracting the asymptotic form of the metric in order to compute M, n. Particularly difficult is n as it is constructed from the difference of two relatively large quantities [25]. Here we compute both n and M using the first law and Smarr formula from T and S, which are conveniently determined from the metric near the horizon.

6D Results.—We now discuss the behavior of 6D localized black holes, and the evidence that they join to the nonuniform branch. First we consider the geometry of the horizon.

We may embed the spatial horizon geometry as a surface of revolution in 5D Euclidean space. This is illustrated for several solutions in Fig. 1. The intrinsic geometry of the embedded surface is the same as that of the spatial sections of the 6D horizons. Note that for the black holes we also include the exposed rotational symmetry axis in the embedding plots. The Euclidean coordinate along the surface of revolution, X, should be thought of as periodic with $-X_{\rm max} < X < X_{\rm max}$ being the fundamental domain plotted. Then $X_{\rm max}$ gives a coordinate invariant measure of the "size" of the geometry near the symmetry axis, and we plot this in Fig. 2.

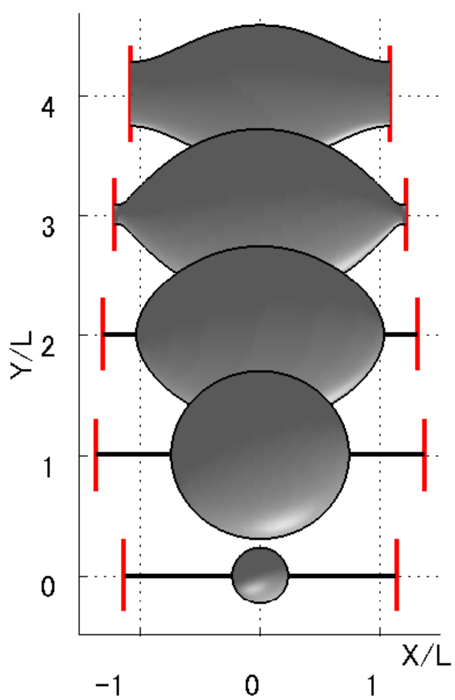

FIG. 1 (color online). Embeddings of the spatial horizon geometry of various 6D BHs and NUSs in 5D Euclidean space (suitably projected onto the page). For BH solutions we include the exposed symmetry axis in the embedding. The red vertical lines are to be periodically identified, generating the compact dimension.

For nonuniform solutions we may compute the maximum and minimum radii of the horizon $R_{\rm max,min}$. Analogously, for the black holes we have $R_{\rm eq}$, the equatorial radius of the horizon, and $L_{\rm axis}$, the proper length along the exposed symmetry axis. These are plotted in Fig. 3.

From these graphs we see firstly that for increasing n the equatorial radius of the black holes reaches a maximum and then starts to decrease, implying that there is indeed a maximum size localized solution that can fit. Second, from the horizon geometry it is plausible that the nonuniform and black hole branches merge around $n/n_{\rm crit} \simeq 0.55$, where the value of $R_{\rm eq}$ appears to tend to $R_{\rm max}$, and $X_{\rm max}$ is consistent with an extrapolation that agrees between the branches for this value of $n/n_{\rm crit}$.

Now we consider thermodynamic quantities. In Figs. 4 and 5 we plot the entropy, temperature, and mass of the solutions against n. Mirroring the behavior of $R_{\rm eq}$, we see the entropy and mass of the black holes reach a maximum and then decrease with increasing n. From this thermodynamic data we again clearly see evidence the branches merge.

Note that in our previous work [25] constructing the 6D black holes, the solutions were only found in the regime where $R_{\rm eq}$ and the mass were increasing with n, and hence it was not clear that the two branches could unify.

For very small masses the localized black holes are entropically favored, and for very large masses only the uniform strings exist. In Fig. 6 we plot the entropy against mass for the 3 branches. We see uniform strings become entropically favored, for a given mass, at masses above that of the critical uniform string, but below that of the maximum mass localized black hole.

Thus we see that the branches appear consistent with merger at $n/n_{\rm crit} \simeq 0.55$. Let us now assume that this

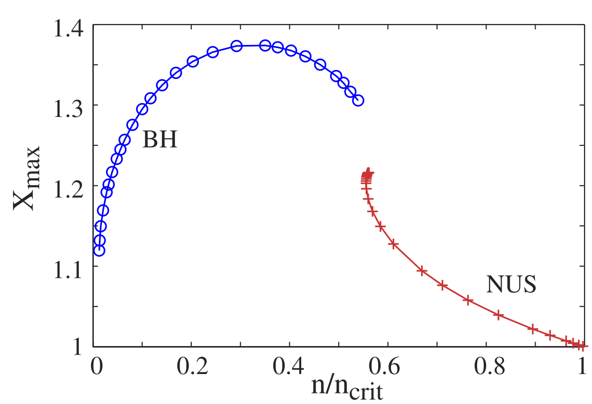

FIG. 2 (color online). Plot of $X_{\rm max}$ for 6D NUSs and BHs, consistent with merger of the branches at $n/n_{\rm crit} \simeq 0.55$.

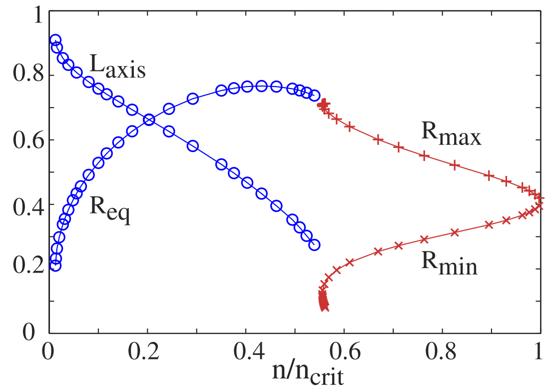

FIG. 3 (color online). Plot of horizon geometric quantities for 6D solutions. Branches are consistent with a topology changing merger where both L_{axis} and $R_{\min}$ go to zero, and R_{eq} tends to $R_{\max}$. All solutions have $L = 1$.

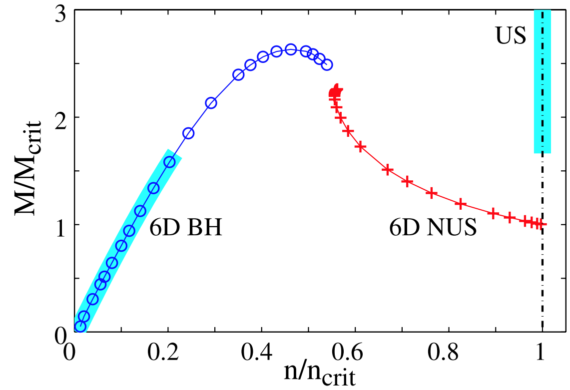

FIG. 5 (color online). Mass against n for 6D solutions. The highlighting indicates which branch is entropically favored for a given mass (see Fig. 6).

occurs via a conical transition. We can then measure how far from the transition point the solutions are by estimating the geometric resolution of the cone. For the nonuniform strings this is given by the minimal radius of the horizon, $R_{\min}$, which for the most nonuniform string found in [22] was $R_{\min} \simeq 0.08$ in our units where $L = 1$. The resolution for the black hole is given by the proper distance of exposed symmetry axis, L_{axis}. For these 6D solutions the smallest resolution found was $L_{\text{axis}} \simeq 0.27$. Therefore the most nonuniform strings found are still considerably closer to the assumed transition point than the most extreme localized black holes found here. For these nonuniform solutions the emergence of the cone geometry has been numerically demonstrated [24]. Repeating this for our new black hole solutions does indeed show an increase in curvature on the axis consistent with an emerging cone, but as the solutions are "further" from the transition point, this increase in curvature is still not large enough to be clearly distinguished from the background curvature of the black hole geometry. Thus while our new 6D black hole data are very suggestive the solution branches merge, they still cannot confirm that the detailed merger from the black hole side is via a conical transition. Indeed in Kol's original picture [10] we note that the cone may act only as an approximate local model for the merger, and the detailed behavior very close to the point where the horizon pinches off may have a complicated behavior that cannot necessarily be thought of as being smoothly resolvable.

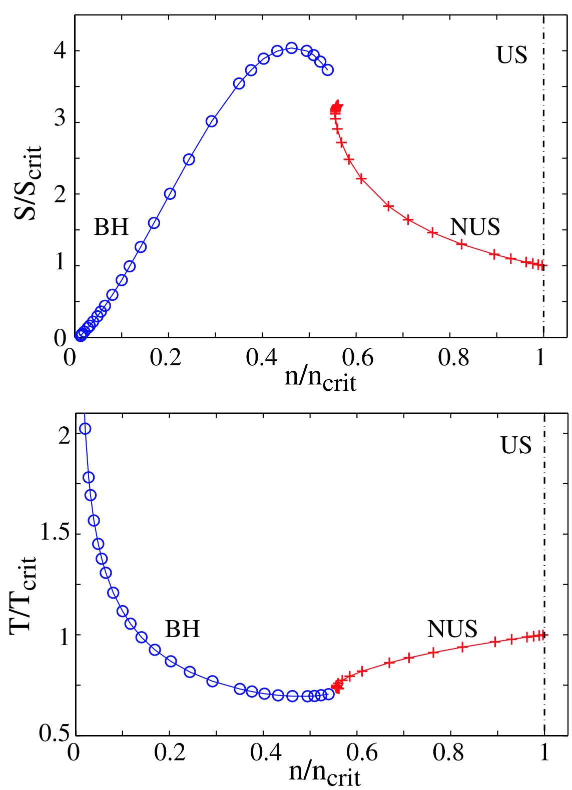

FIG. 4 (color online). Entropy and temperature for 6D solutions.

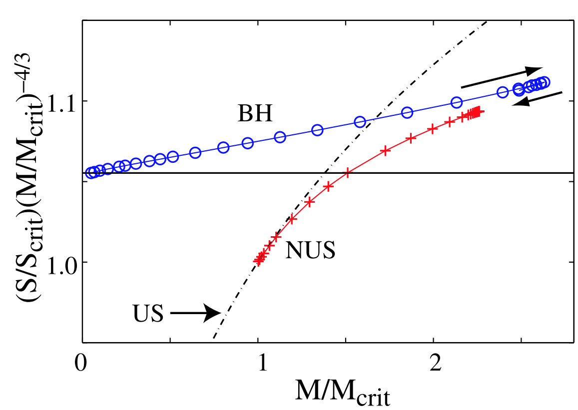

FIG. 6 (color online). Plot of $SM^{-4/3}$ against mass for the 6D solutions. 6D Schwarzschild behavior is $S \propto M^{4/3}$ so represents a horizontal line. We see the localized BHs are entropically favored for a given mass, for $M < 1.7M_{\text{crit}}$. Above this mass the uniform strings are favored. The NUSs are never dominant.

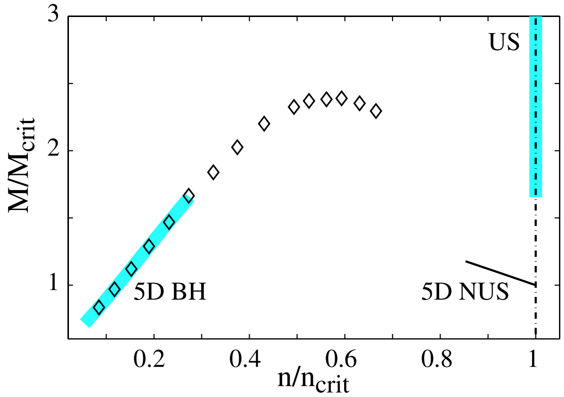

FIG. 7 (color online). Mass against n for new 5D localized BHs. Highlighting indicates entropically favored solution at a given mass.

5D Results.—We now briefly discuss the 5D black hole solutions. All quantities behave in an analogous manner to their counterparts for the 6D solutions. Here we have simply plotted the mass against n in Fig. 7. Note the mass increases past the critical uniform string mass with increasing n (extending the previous numerical solutions in 5D [26]), reaching a maximum and then decreases, again presumably to join the nonuniform branch [36].

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