Atmospheric Turbulence and Orbital Angular Momentum of Single Photons for Optical Communication

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The effects of propagation through random aberrations on coherence for single-photon communication systems based on orbital angular momentum states are quantified. A rotational coherence function is derived which leads to scattering equations for azimuthal modes of different orbital angular momentum states. The effect on a single-photon communication system is quantified using the channel capacity. The work shows that the decoherence effect of atmospheric turbulence on such systems is important even for weak turbulence.

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Photons carry both spin and orbital angular momentum, both of which offer practical realization of quantum digits and the possibility of secure single-photon optical communication [1]. The spin angular momentum is associated with the polarization, which, for a given propagation direction, is described by a two-dimensional state and can be used as the basis of a quantum binary digit or qubit. Secure quantum key distribution over a free-space link using polarization states has been demonstrated [2]. The orbital angular momentum (OAM) is associated with the spatial distribution of the wave function and there are in principle an infinite number of OAM eigenstates available. This offers the possibilities of realizing arbitrary base-N quantum digits and of higher capacity optical communication and quantum cryptography corresponding to a base-N digit per single photon [3], and there is considerable interest in free-space or line-of-sight communications due to the spatial-image nature of OAM. Recently, an interferometric sorter has been demonstrated which could be extended to measure the OAM of a photon with, in principle, 100% efficiency [4]. However, the spatial-image nature of OAM suggests that it may be susceptible to spatial aberrations such as arise in atmospheric turbulence [5]. Refractive index fluctuations in the atmosphere associated with turbulence give rise to random phase aberrations on a propagating optical beam. The resulting scattering at optical wavelengths is characterized as small-angle forward scattering and the effect on the polarization state of the beam, and thus the spin states of photons, is very small even for relatively strong atmospheric turbulence aberrations. However, the effect on OAM cannot be dismissed. In this Letter, I address the effect of random phase aberrations on OAM of single photons and show that even weak aberrations can have a severe impact on the fundamental operation of systems using OAM.

I present a semiclassical approach to obtain the probability distribution for measurements of the orbital angular momenta of photons after propagation through random phase aberrations. This leads naturally to the description of the spatial coherence of the wave in terms of a *rotational coherence function*, from which relations for the effective scattering between different azimuthal modes associated with OAM are obtained. The OAM measurement probabilities relate directly to the probabilities of obtaining correct or incorrect measurements of a transmitted "symbol" in a communication link. I use this approach to evaluate the OAM scattering for Kolmogorov atmospheric phase aberrations and to estimate the theoretical performance limits this implies for communication links based on OAM of single photons.

Consider a paraxial beam that initially has a transverse spatial wave function corresponding to an eigenstate of OAM. Without loss of generality, we write the OAM eigenfunctions as

$$\varphi_{p,l}(r,\theta) = R_p(r)\frac{\exp(il\theta)}{\sqrt{2\pi}}, \quad (1)$$

with eigenvalues $l_z = l\hbar$, where l is the azimuthal mode order, and $R_p(r)$ comprise a radial basis set (normalized with weight r). As the beam propagates, it is scattered by random refractive index inhomogeneities, the effect of which is a phase aberration that perturbs the complex amplitude of the wave (Fig. 1). The resulting wave $\Psi(r,\theta,z)$ is a superposition of eigenstates and the conditional probability of obtaining a measurement of the OAM

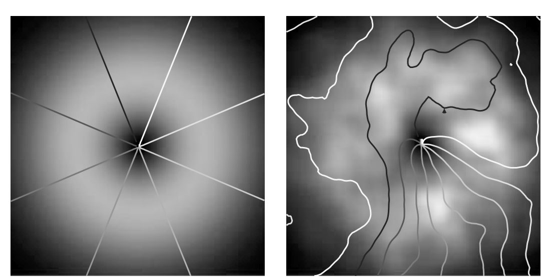

FIG. 1. Example intensity (gray scale) and phase map ($\pi/4$-spaced contours) of a pure LG_1^0 beam (left) and the same beam with aberrations caused by propagation through Kolmogorov turbulence (right).

of a photon $l_z = l\hbar$ is obtained by summing the probabilities associated with that eigenvalue,

$$P(l|\Psi) = \sum_p |a_{p,l}(z)|^2. \tag{2}$$

Substituting for the superposition coefficients $a_{p,l}(z)$, which are given in the usual way as $a_{p,l}(z) = \langle \varphi_{p,l}|\Psi(r, \theta, z)\rangle$, and rearranging, this becomes

$$P(l|\Psi) = \iint \iint \Psi^*(r', \theta', z)\Psi(r, \theta, z)\sum_p R_p^*(r)R_p(r') \times \frac{\exp[-il(\theta - \theta')]}{2\pi} r'dr'd\theta' rdrd\theta.$$

Using the completeness of the radial basis, $\sum_p R_p^*(r)R_p(r') = \delta(r, r')/r$, and integrating over r', gives an expression that is independent of the initial choice of radial basis,

$$P(l|\Psi) = \iint \int \Psi^*(r, \theta', z) \times \Psi(r, \theta, z)rdr\frac{\exp[-il(\theta - \theta')]}{2\pi} d\theta' d\theta.$$

The aberrations are random, so the probability that a measurement of the OAM will yield a value $l_z = l\hbar$ is found by taking the ensemble average, thus,

$$\begin{aligned} P(l) &= \langle P(l|\Psi)\rangle \\ &= \iint \int \langle \Psi^*(r, \theta', z) \\ &\quad \times \Psi(r, \theta, z)\rangle rdr\frac{\exp[-il(\theta - \theta')]}{2\pi} d\theta' d\theta, \end{aligned} \tag{3}$$

where $\langle\cdot\rangle$ denotes the ensemble average. We shall refer to the ensemble average term in the integrand as the *rotational field correlation* at radius r. Assuming the statistics of the turbulence aberrations to be isotropic and since the beam profile at launch was rotationally symmetric, then the rotational field correlation can be written as

$$C_\Psi(r, \Delta\theta, z) = \langle \Psi^*(r, 0, z)\Psi(r, \Delta\theta, z)\rangle, \tag{4}$$

where $\Delta\theta = \theta - \theta'$, and (3) reduces to

$$P(l) = \iint C_\Psi(r, \Delta\theta, z)rdr \exp(-il\Delta\theta)d\Delta\theta. \tag{5}$$

Thus, the OAM probability distribution is determined by the circular harmonic transform (Fourier series expansion) of the rotational field correlation, which is a function of the spatial coherence properties of the wave. (See, for example, [6] for a consideration of finite temporal coherence in vortex fields.) It can be considered a generalization of the Fourier relationship between azimuthal spatial distribution and OAM [7].

We now need to consider how atmospheric turbulence influences the rotational field correlation. The rotational field correlation is a function of the second order spatial statistics of the complex amplitude fluctuations. Although there exists a significant body of work on analytical modelling for these second order statistics for plane wave and spherical wave geometries, the problem is highly nontrivial for general beam geometries, and analytical modelling has been described only for limited geometries and beam shapes [8,9]. We shall therefore restrict ourselves to considering the weak turbulence regime, in which intensity fluctuations or scintillation arising from the turbulent phase aberrations are sufficiently small that they can be neglected. In this approximation, the cumulative effect of the turbulence over the propagation path can be considered as a pure phase perturbation on the beam at the output plane z. The complex amplitude of the beam is then given by

$$\Psi(r, \theta, z) = R(r, z)\frac{\exp(il_0\theta)}{\sqrt{2\pi}} \exp[i\phi(r, \theta)], \tag{6}$$

where $\phi(r, \theta)$ is the phase perturbation and l_0 is the initial OAM quantum number of the unperturbed beam. Substituting into the rotational field correlation (4) and simplifying gives

$$C_\Psi(r, \Delta\theta, z) = |R(r, z)|^2 C_\phi(r, \Delta\theta)\frac{\exp(il_0\Delta\theta)}{2\pi}, \tag{7}$$

where $C_\phi(r, \Delta\theta)$ is given by

$$C_\phi(r, \Delta\theta) = \langle \exp\{i[\phi(r, \Delta\theta) - \phi(r, 0)]\}\rangle. \tag{8}$$

We shall refer to $C_\phi(r, \Delta\theta)$ as the *rotational coherence function* of the phase perturbations at radius r. The probabilities for the angular momentum measurements (5) become

$$P(l) = \int_0^\infty |R(r, z)|^2 r\Theta(r, l - l_0)dr, \tag{9}$$

where $\Theta(r, \Delta l)$ is the circular harmonic transform of the rotational coherence function,

$$\Theta(r, \Delta l) = \frac{1}{2\pi}\int_0^{2\pi} C_\phi(r, \Delta\theta) \exp[-i\Delta l\Delta\theta]d\Delta\theta. \tag{10}$$

Equation (9) can be viewed as a scattering equation for optical power between different OAM states. $\Theta(r, \Delta l)$ are the scattering coefficients between azimuthal modes for optical power in an annulus of radius r. The OAM probability distribution is obtained by averaging over the radial power distribution of the beam $|R(r, z)|^2 r$. Since the scattering coefficients depend only on the difference $\Delta l = l - l_0$, for a given radial beam profile $R(r, z)$ the resultant OAM spreading is independent of the initial eigenvalue l_0 and therefore we have probabilities for correct or incorrect measurements of a transmitted digit in an OAM communication link that are independent of the digit value. (This disagrees with [5], which argues that $l = 1$ beams are less susceptible due to the stability of an $l = 1$ singularity. However, it is the wave function across the beam rather than a singularity at its center that determines the OAM distribution.)

All that remains is to find an expression for the rotational coherence function and the corresponding scattering coefficients. Assuming the refractive index fluctuations to be a

Gaussian random process, which allows the standard result $\langle \exp(ix)\rangle = \exp(-\frac{1}{2}\langle |x|^2\rangle)$ to be used, then the rotational correlation function can be written

$$C_\phi(r, \Delta\theta) = \exp\left[-\frac{1}{2}D_\phi\left(\left|2r\sin\left(\frac{\Delta\theta}{2}\right)\right|\right)\right], \quad (11)$$

where $D_\phi(|\Delta x|) = \langle|\phi(x) - \phi(x + \Delta x)|^2\rangle$ is the phase structure function of the aberrations. For Kolmogorov turbulence phase aberrations, $D_\phi(\Delta x) = 6.88(\Delta x/r_0)^{5/3}$ where r_0 is the Fried parameter [10]. Thus, for Kolmogorov turbulence phase statistics, the rotational coherence function at radius r is

$$C_\phi(r, \Delta\theta) = \exp\left[-6.88 \times 2^{2/3}\left(\frac{r}{r_0}\right)^{5/3}\left|\sin\left(\frac{\Delta\theta}{2}\right)\right|^{5/3}\right]. \quad (12)$$

In Fig. 2 the first few OAM scattering coefficients $\Theta(r, \Delta l)$ are plotted against r/r_0. The Fried parameter r_0 corresponds approximately to the spatial coherence length of the aberrations. For $r \ll r_0$, the effects of the phase aberrations are weak and the OAM scattering is small, but they increase rapidly as r becomes comparable to r_0. Thus the performance of a communication system will depend strongly on the beam size relative to r_0. Narrow beams with the optical power concentrated close to the optical axis will be less affected.

The free-space Laguerre–Gaussian (LG) beam modes (LG_l^p) have transverse wave functions which are eigenfunctions of OAM and have therefore been the subject of considerable interest recently for single-photon communications using OAM states and for possible implementation of base-N quantum digits [3]. It is therefore interesting to see how the OAM of such beams are affected by turbulence. The OAM probabilities for various LG beams propagating through Kolmogorov turbulence were evaluated using (9), substituting $|R(r, z)|^2$ with the LG radial intensity profiles. Figure 3 plots probabilities of different OAM measurements for a single beam mode LG_1^0. For different order beam modes, the effect of the phase perturbations depends on the radial power distribution in the beam, which is characterized by the second radial moment of the intensity and which for an LG_l^p beam is

$$\langle r^2\rangle = \int_{r=0}^{\infty} R_{l,p}(r) r^2 r dr = (2p + l + 1)b^2, \quad (13)$$

giving a characteristic rms beam radius $r_{p,l} = b\sqrt{2p + l + 1}$. Figure 4 plots the probabilities for obtaining the original OAM eigenvalue scaled against this rms beam radius for several different LG_l^p beams. Scaled to the rms beam radius, the behavior for different beam modes is very similar.

The OAM probabilities can now be used to quantify the effects on a communication link using the channel capacity [11]. The channel capacity is defined as

$$C = \max[H(x) - H(x|y)], \quad (14)$$

where $H(x)$ is the entropy of the source, and $H(x|y)$ its conditional entropy (or the equivocation) given received data y,

$$H(x) = -\sum_{x_i} P(x_i)\log P(x_i), \quad (15)$$

$$H(x|y) = -\sum_{y_j}\sum_{x_i} P(x_i, y_j)\log P(x_i|y_j), \quad (16)$$

and where $P(x_i)$ is the probability for the transmitted signals $\{x_i\}$, $P(x_i, y_j)$ is the joint probability for $\{x_i\}$ and received signals $\{y_j\}$ and $P(x_i|y_j)$ is the conditional probability for $\{x_i\}$ given $\{y_j\}$.

Consider a line-of-sight communication link using OAM eigenstates in the range $l = -L, \ldots, L$ for different signal values. The source entropy is maximized when the probabilities for the transmitted states are uniform thus $P(x_i) = 1/(2L + 1)$, giving for the source entropy $H(x) = \log(2L + 1)$.

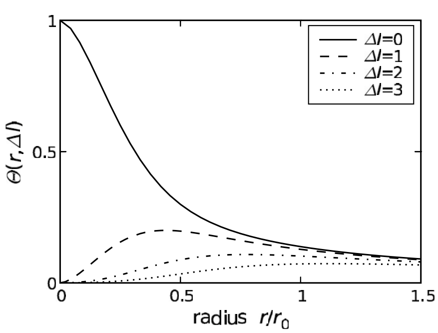

FIG. 2. OAM scattering coefficients $\Theta(r, \Delta l)$ between modes l_0 and $l_0 \pm \Delta l$ for an annulus of radius r in a beam for aberrations with Fried parameter r_0. The $\Delta l = 0$ plot corresponds to the unscattered light.

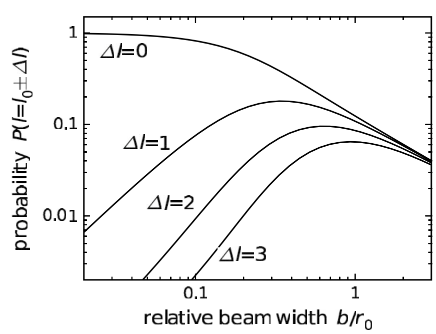

FIG. 3. Probabilities of obtaining different OAM measurements $P(l = l_0 \pm \Delta l)$ for a LG_1^0 beam plotted against the ratio of the Gaussian beam width parameter b to the Fried parameter r_0. The probability of obtaining the original eigenvalue ($\Delta l = 0$), i.e., the correct signal, decreases rapidly as the Fried parameter becomes comparable to the beam width parameter. There is a corresponding increase in the probabilities of obtaining OAM measurements different from the initial eigenvalue ($\Delta l \neq 0$), with those corresponding to adjacent azimuthal modes increasing most rapidly.

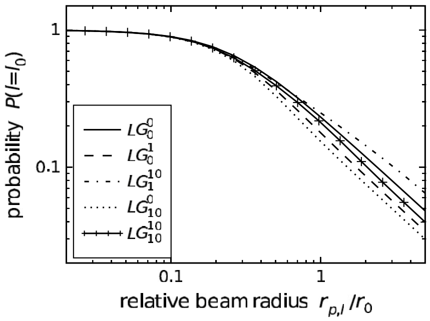

FIG. 4. Probabilities of obtaining the original eigenvalue $P(l = l_0)$ for several different LG beam modes plotted against the ratio of the rms beam radius $r_{p,l}$ to the Fried parameter.

The OAM of photons measured at the receiver are not guaranteed to be those of the transmitted photons due to scattering. This will give a nonzero value for the equivocation. The set of receiver signal values $\{y_j\}$ comprises measurements of OAM eigenvalues in the range $l = -L, \ldots, L$, and an extra signal value corresponding to "lost" photons (i.e., those photons scattered outside the original range of OAM). The conditional probabilities for the receiver signals $P(y_j|x_i)$ are simply the OAM measurement probabilities that we have already found [Eq. (9)] and the equivocation can be rewritten in terms of them as

$$H(x|y) = -\sum_{ij} P(x_i)P(y_j|x_i)\left[\log P(y_j|x_i) - \log \sum_i P(y_j|x_i)\right].$$

Figure 5 plots the channel capacities for systems using different numbers of LG beam modes LG_l^0 for $l = -L, \ldots, L$, for Kolmogorov turbulence. The channel capacity decreases rapidly as the turbulent aberrations increase. For a three-state system with $L = 1$, the channel capacity is reduced to below that for an ideal binary state channel for $b/r_0 > 0.1$. For example, with moderate ground-level turbulence strength $C_n^2 = 10^{-14}\ \mathrm{m}^{-2/3}$ and wavelength $\lambda = 1\ \mu\mathrm{m}$ this corresponds to a mere $z = 200$ m propagation path for a beam with $b = 1$ cm (which is a narrow beam, comparable to the Fresnel length $\sqrt{\lambda z}$, chosen to give a "best-case" scenario).

The results show that the effects even of weak aberrations are significant, and this poses a considerable problem for communication systems based on OAM. The key parameter determining the magnitude of the effect is the beam width relative to the coherence scale of the aberrations (the ratio b/r_0.) The narrower the beam, the less is the OAM scattering, although, diffraction places a lower limit on the practical choice of beam width to around the Fresnel length. This behavior is in contrast to intensity modulation systems for which scintillation is more important, the effects of which can be reduced by using wider beams. Higher-order mode beams are even more susceptible due to their wider intensity distributions. The phase aberrations nature of the problem suggests adaptive optics may be of considerable benefit [12]. Although the theory has been applied here to Kolmogorov atmospheric statistics, this assumption is not critical, and one would expect comparable results for other atmospheric models. The rotational coherence and scattering analysis presented here is quite general and could be applied to other sources of perturbations. Indeed, the demonstrated sensitivity of OAM states to phase perturbations means that this will be an important consideration generally for systems using OAM. Moreover, since the key attribute is the spatial-image nature of the information, other spatial-image schemes, for example Hermite-Gaussian modes, will be similarly affected.

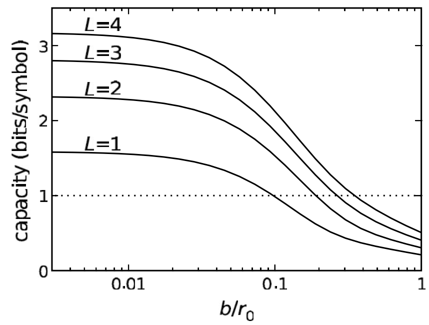

FIG. 5. Plots of the channel capacity for a communication link employing OAM states of single photons for different numbers of OAM states $l = -L, \ldots, L$ using LG beams LG_l^p. The dashed line shows the capacity for a polarization-based channel (two-state) for comparison.

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[1] L. Allen *et al.*, Phys. Rev. A **45**, 8185 (1992).
[2] W. T. Buttler *et al.*, Phys. Rev. Lett. **81**, 3283 (1998); J. G. Rarity, P. M. Gorman, and P. R. Tapster, Electron. Lett. **37**, 512 (2001).
[3] A. Mair *et al.*, Nature (London) **412**, 313 (2001); G. Molina-Terriza, J. P. Torres, and L. Torner, Phys. Rev. Lett. **88**, 013601 (2002).
[4] J. Leach *et al.*, Phys. Rev. Lett. **88**, 257901 (2002).
[5] G. Gibson *et al.*, Opt. Express **12**, 5448 (2004).
[6] G. A. Swartzlander and J. Schmit, Phys. Rev. Lett. **93**, 093901 (2004).
[7] S. Franke-Arnold *et al.*, New J. Phys. **6**, 103 (2004).
[8] Albert D. Wheelon, *Electromagnetic Scintillation II. Weak Scattering* (Cambridge University Press, Cambridge, England, 2003).
[9] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation Through Random Media* (SPIE, Washington, 1998).
[10] D. L. Fried, J. Opt. Soc. Am. **56**, 1372 (1966).
[11] C. E. Shannon, Bell Syst. Tech. J. **27**, 379 (1948).
[12] C. Paterson, in *Optics in Atmospheric Propagation and Adaptive Systems VII*, edited by John D. Gonglewski and Karin Stein, SPIE Proceedings Vol. 5572 (SPIE-International Society for Optical Engineering, Bellingham, WA, 2004), p. 187.