Hilbert's 17th Problem and the Quantumness of States

J. K. Korbicz,[1] J. I. Cirac,[2] Jan Wehr,[3] and M. Lewenstein[1]

[1]*Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany*

[2]*Max-Planck Institut für Quantenoptik, Hans-Kopfermann Str. 1, D-85748, Garching, Germany*

[3]*Department of Mathematics, University of Arizona, 617 N. Santa Rita Ave., Tucson, Arizona 85721-0089, USA*

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A state of a quantum system can be regarded as classical (quantum) with respect to measurements of a set of canonical observables if and only if there exists (does not exist) a well defined, positive phase-space distribution, the so called Glauber-Sudarshan P representation. We derive a family of classicality criteria that requires that the averages of positive functions calculated using P representation must be positive. For polynomial functions, these criteria are related to Hilbert's 17th problem, and have physical meaning of generalized squeezing conditions; alternatively, they may be interpreted as nonclassicality witnesses. We show that every generic nonclassical state can be detected by a polynomial that is a sum-of-squares of other polynomials. We introduce a very natural hierarchy of states regarding their degree of quantumness, which we relate to the minimal degree of a sum-of-squares polynomial that detects them.

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In recent years there has been a lot of interest in classifying states of quantum systems with respect to their quantum nature. In particular, the problem of characterizing entangled states has attracted a lot of attention [1] because of its vital importance for quantum information processing. One of the aspects of this problem concerns nonlocality of quantum mechanics and violations of Bell-like inequalities [2] and the existence of local hidden variable models. The problem of the existence of a classical probabilistic description of quantum states of a single system has, however, a longer history and can be traced back to the seminal papers of Glauber and Sudarshan [3].

Let us consider a harmonic oscillator Hilbert's space, and fix the canonical creation and annihilation operators, a, $a^\dagger$. In the Refs. [3] it was shown that any state ϱ has a P representation, i.e., can be uniquely put into a form diagonal in coherent states $|\alpha\rangle$:

$$\varrho = \int d^2\alpha P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|, \tag{1}$$

where $\alpha = x + iy$, $d^2\alpha = dxdy$. The integration with P in (1) is understood in the distributional sense [4]. Hermiticity and normalization of ϱ imply that $P^* = P$ and $\int d^2\alpha P(\alpha, \alpha^*) = 1$, while positivity implies that $\int d^2\alpha P(\alpha, \alpha^*)|\langle\alpha|\psi\rangle|^2 \geq 0$ for every ψ.

A state of a quantum system is classical with respect to measurements of a given set of canonical observables if and only if the Glauber-Sudarshan P representation is a well defined, positive phase-space distribution [5]. Mathematically speaking, in such a situation P defines a probabilistic measure μ on the phase-space $\mathbb{R}^2$ through

$$\mathbb{R}^2 \supset \Omega \mapsto \mu(\Omega) := \int_\Omega P d^2\alpha \geq 0. \tag{2}$$

Statistical properties of states with the positive P representation are as those of the classical statistical ensembles, described by the measure μ; that explains why such states are called classical. However, the class of allowed P's is larger than that [4], and there exist nonclassical states (such as squeezed, or Fock states) for which the integral (2) does not always exist, or attains negative values.

In this Letter we derive a family of classicality criteria that require that averages of positive functions calculated using P representation must be positive. For polynomial functions, these criteria are related to Hilbert's 17th problem, which in its simplest form states that not every positive semidefinite polynomial must be a sum-of-squares (SOS) of other polynomials [6]. Our criteria have physical meaning of generalized squeezing conditions, and may be interpreted as nonclassicality witnesses (in analogy to entanglement witnesses, [7]). We show that every generic nonclassical state can be detected by a SOS polynomial.

Let us begin by observing that the set of probabilistic measures forms a convex subset of the set of all P's. The extreme points of this set are point concentrated measures $\{\delta^2(\alpha - \beta); \beta \in \mathbb{C}\}$, and the decomposition into these points is unique [8]. Hence the classical states form a generalized simplex Δ. This is the general feature of sets of probabilities for classical systems [9]. Therefore, geometrically the problem of distinguishing between classical and nonclassical states amounts to the operational description of the simplex of measures Δ in the space of all P distributions. We note that one encounters closely related problems in the study of quantum entanglement in multipartite systems (see [7] and references therein), with the difference being that the convex subset of classically correlated states is not a simplex.

The solution of such stated problem was recently proposed by Richter and Vogel [10]. They studied the characteristic function of P, i.e., its Fourier transform

$$\hat{P}(\xi) := \int d^2\alpha P(\alpha, \alpha^*) e^{2i(\xi_i x - \xi_r y)} = \mathrm{tr}\{\varrho : W(\xi) :\}, \tag{3}$$

where $\xi = \xi_r + i\xi_i$ and $W(\xi) = e^{\xi a^\dagger - \xi^* a}$ is the Weyl op-

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erator. In what follows the hat, $\hat{}$, will always denote Fourier transform. The criterion detecting positive measures is then provided by Bochner's theorem [11].

Theorem.—$\hat{P}$ is a Fourier transform of a probabilistic measure if and only if $\hat{P}$ is of positive type, i.e., for each number n and all possible sets $\xi_1, \ldots, \xi_n \in \mathbb{R}^2$ the $n \times n$ matrix $\hat{P}_{ij} := \hat{P}(\xi_i - \xi_j)$ is positive semidefinite (PSD).

The further test of $\hat{P}_{ij}$ being PSD for fixed $\xi_1, \ldots, \xi_n$ is carried out using determinant criterion: a $n \times n$ matrix is PSD if and only if determinants D_k, $k = 1 \ldots n$ of all of the principal submatrices are nonnegative. This finally leads to the hierarchy of conditions: a state ϱ is nonclassical if and only if there exist $k > 2$ (for $k = 1$, $D_1 = 1$ due to normalization) and points $\xi_1, \ldots, \xi_k$ such that $D_k < 0$.

Our solution of the classicality-quantumness problem follows also from the Bochner's theorem. Note that the condition for $\hat{P}$ to be a function of positive type can be equivalently rewritten as: $\hat{P}$ is a function of positive type if and only if for all $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\int d^2\alpha d^2\beta \chi(\alpha)^* \hat{P}(\alpha - \beta)\chi(\beta) \geq 0$, where $\mathcal{D}(\mathbb{R}^2)$ is a space of smooth test functions with compact support [11]. Using the convolution theorem the last integral is equal to $\int d^2\alpha P(\alpha, \alpha^*)|\hat{\chi}(-\alpha)|^2$. From Fourier transform theory, $\hat{\chi}$ can be analytically continued to a function from $\mathcal{Z}(\mathbb{C}^2)$, the space of entire functions, satisfying specific bonds [12], and every element of $\mathcal{Z}(\mathbb{C}^2)$ is of that form [11,12]. Hence we obtain the criterion for classicality [13]:

Theorem.—P defines a probabilistic measure if and only if

$$\forall f \in \mathcal{Z}(\mathbb{C}^2), \qquad \int d^2\alpha P\, |f_{\mathbb{R}}|^2 \geq 0, \tag{4}$$

where $f_{\mathbb{R}}$ denotes the restriction of f to $\mathbb{R}^2$.

Our approach offers new insights into the problem, and connects it to the methods used in the study of separability. From (4) we obtain that a state ϱ is nonclassical if and only if there exists a test function $f \in \mathcal{Z}(\mathbb{C}^2)$ such that $\int d^2\alpha P\, |f_{\mathbb{R}}|^2 < 0$. Since $f_{\mathbb{R}}$ is real-analytic this condition can be rewritten as

$$\mathrm{tr}\{\varrho{:}|f_{\mathbb{R}}(a, a^\dagger)|^2{:}\} < 0, \tag{5}$$

implying that the state is nonclassical if and only if there exists an observable $:|f_{\mathbb{R}}(a, a^\dagger)|^2:$ detecting it. Geometrically, the condition $\int d^2\alpha P|f_{\mathbb{R}}|^2 = 0$ defines a hyperplane in the set of all P distributions and hence a state is nonclassical if and only if there is a hyperplane separating it from the simplex Δ. This is essentially the same approach as the one used in the theory of entanglement witnesses [14,15]. Therefore, we propose to call the observable from expression (5) nonclassicality witness. The above approach can be generalized if we allow the test functions f to depend on the state ϱ in question. Then, the observable in Eq. (5) becomes a nonlinear function of the state, and may be termed a nonlinear nonclassicality witness (compare [16]).

In the current Letter we restrict the class of investigated states ϱ to those, for which P can be evaluated on an arbitrary real polynomial of x, y. The vector space of such polynomials will be denoted by $\mathbb{R}[x, y]$. Since any polynomial can be represented as a Fourier transform of appropriate sum of derivatives of the Dirac's delta function, the sufficient condition for that is that $\hat{P}$ is a smooth function. We denote the space of such P's by $\mathcal{P}$.

Note, that since the test function f appearing in (4) is entire, it can be almost uniformly approximated by a sequence of complex polynomials on $\mathbb{C}^2$. Hence $|f_{\mathbb{R}}|^2 = \lim_{N\to\infty}(u_N^2 + v_N^2)$ for some $u_N, v_N \in \mathbb{R}[x, y]$. The almost uniform convergence on the real plane allows us to interchange integration and taking the limit in (4) [12]. This leads to the main theorem of the present Letter:

Theorem.—A state ϱ with $P \in \mathcal{P}$ is classical if and only if for every polynomial $v \in \mathbb{R}[x, y]$

$$\int d^2\alpha P\, v^2 = \mathrm{tr}\{\varrho{:}v(a, a^\dagger)^2{:}\} \geq 0. \tag{6}$$

In fact the criterion (6) has already been used for a long time for detecting some important classes of quantum states. There are two examples of the application of (6) known to the authors. The first one is the test for higher order quadrature squeezing [17]: ϱ is squeezed to the order $2k$ if there exists a phase $\phi \in [0, 2\pi)$ such that

$$\sum_{l=0}^{k-1} \frac{1}{2^l} \frac{(2k)!}{l![2(k-l)]!} \langle{:}(\Delta E_\phi)^{2(k-l)}{:}\rangle < 0, \tag{7}$$

with $E_\phi := a e^{-i\phi} + a^\dagger e^{i\phi}$, $\Delta E_\phi := E_\phi - \langle E_\phi \rangle$, and the averages taken with respect to ϱ. Obviously, (7) has the form of a violation of (6) (we can always substitute v^2 there with finite sums of such terms) with the polynomial

$$w_{2k}(x, y; \phi) := \sum_{l=0}^{k-1} \frac{1}{2^l} \frac{(2k)!}{l![2(k-l)]!} [d_\phi(x, y)]^{2(k-l)}, \tag{8}$$

where

$$d_\phi(x, y) := 2[x - \langle \tfrac{a+a^\dagger}{2} \rangle] \cos\phi + 2[y - \langle \tfrac{a-a^\dagger}{2i} \rangle] \sin\phi. \tag{9}$$

The witness w_{2k} depends on the tested state ϱ and hence is a nonlinear witness.

The second example is the test for sub-Poissonian statistics of $a^\dagger a$ (number squeezing): ϱ is number squeezed if $\langle{:}(\Delta a^\dagger a)^2{:}\rangle < 0$. The corresponding nonlinear witness is

$$w_P(x, y) := (x^2 + y^2 - \langle a^\dagger a \rangle)^2. \tag{10}$$

Note that both nonlinear witnesses (8) and (10) are optimal in the sense that they are zero on the extreme points of Δ, as for any $|\alpha\rangle$ all the moments of normally ordered deviations vanish.

From (6), we observe that for any $v \in \mathbb{R}[x, y]$, v^2 is positive semidefinite (PSD), and so is every polynomial which is a sum of such terms (we call such polynomials SOS polynomials). One may ask if the converse is also true, i.e., if every PSD polynomial is SOS?

This problem has been known in mathematics under the name of Hilbert's 17th problem. The answer is, quite surprisingly, negative: there are PSD polynomials which are not SOS [6]. For the case of 3 variables this happens for a degree $m \geq 6$. However, the explicit examples of PSD, but not SOS polynomials are rare and were found quite lately. Note that the connection between Hilbert's 17th problem and separability was established in [18].

In light of the theorem (6), out of all PSD polynomials, SOS polynomials are sufficient to detect nonclassical states among the states with $P \in \mathcal{P}$. To illustrate how the theorem (6) works let us consider a specific example of sixth order Motzkin polynomial which is PSD, but non-SOS:

$$M(x, y, z) := (x^2 + y^2 - 3z^2)x^2y^2 + z^6. \tag{11}$$

Using a method originating from the witness techniques in entanglement study [15], we construct a state ϱ, detected by the polynomial $M(x, y, \pm 1)$ [19].

Out of the four zeros $\{(\pm 1, \pm 1)\}$ of $M(x, y, \pm 1)$ we construct coherent states: $\alpha_1 := 1 + i$, $\alpha_2 := -1 + i$, $\alpha_3 := \alpha_2^*$, $\alpha_4 := \alpha_1^*$. We pick the barycentric point, $\tilde{\varrho}$, of the face $\mathcal{F} := \mathrm{conv}\{\delta(\alpha - \alpha_1), \ldots, \delta(\alpha - \alpha_4)\}$ (conv stands for a convex hull) of the simplex Δ. Note, that the hyperplane defined by the witness $:M(a, a^\dagger, \pm 1):$

$$h_M := \{P \in \mathcal{P}\,;\, \int dxdyP(x, y)M(x, y, \pm 1) = 0\}, \tag{12}$$

contains the face $\mathcal{F} \subset \Delta$ and hence the witness is optimal. Thus, to get the state detected by (11), we mix $\tilde{\varrho}$ with a projector onto an arbitrary vector from its range:

$$\varrho := \frac{1 - \epsilon}{4} \sum_{j=1}^{4} |\alpha_j\rangle\langle\alpha_j| + \epsilon|\psi\rangle\langle\psi|, \tag{13}$$

which for simplicity we choose to be:

$$\psi := \frac{1}{N}(|\alpha_i\rangle + |\alpha_i^*\rangle), \qquad N^2 = 2[1 + e^{-2}\cos(2)]. \tag{14}$$

Here $0 \leq \epsilon \leq 1$ and $i \in \{1, 2, 3, 4\}$ is fixed, but the results presented below do not depend on its particular value. Calculating the average of the polynomial (11) using the expression (6), we obtain $\langle :M(a, a^\dagger, \pm 1): \rangle = (2/N^2)e^{-2}\cos(2)\epsilon$. Since $\cos(2) < 0$, the state (13) is detected by M for $\epsilon > 0$.

As a side remark, we note that the state (13) is also detected by another example of PSD, but non-SOS polynomial—Choi-Lam polynomial $S(x, y, z) := x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$, as $\langle :S(a, a^\dagger, \pm 1): \rangle = -(4/N^2)e^{-2}\sin(2)\epsilon < 0$.

Before we explicitly construct a SOS polynomial detecting (13), let us first examine the physically relevant witnesses (8) and (10). A simple calculation gives that $\langle :(\Delta a^\dagger a)^2: \rangle \geq 0$ for any $0 < \epsilon \leq 1$. Examination of the witnesses w_{2k} is more difficult and we have carried it out only numerically. We checked that up to the 14th order ($k = 7$) all the inequalities (7) are violated for any $\phi \in [0, 2\pi)$ and $0 < \epsilon \leq 1$ and hence (13) is not squeezed up to the order of 14. Apart from that we used a modified version of w_4: $d_{\phi_1}(x, y)^2 d_{\phi_2}(x, y)^2 + 6d_{\phi_3}(x, y)^2$, depending on three angles ϕ_1, ϕ_2, and ϕ_3, also with no success. The question if ϱ has even higher order squeezing is open.

To construct a SOS polynomial detecting ϱ, note that $M(x, y, \pm 1)$ has only four zeros, and hence we can find a second order polynomial with the same zeros, which squared will give us the desired witness. Equivalently, we look for such a SOS witness W that its hyperplane h_W, defined as in (12), contains $\mathcal{F}$. We choose $W(x, y) = (Ax^2 + By^2 + Cxy + Dx + Ey + F)^2$. The condition $W(\pm 1, \pm 1) = 0$ leads to a system of four linear equations for $A, \ldots, F$. Its solution gives a family of witnesses $W_{A,B}(x, y) = (Ax^2 + By^2 - A - B)^2$, where $A^2 + B^2 \neq 0$. The average of $W_{A,B}$ in the state (13) is negative if and only if $\cos(2)[(A + B)^2 - 4A^2] + 4\sin(2)A(A + B) < 0$. As this equation possesses nonzero solutions, for example $A = 0, B \neq 0$, the state ϱ can be detected by a fourth order SOS polynomial.

This seems to be a generic feature, at least for the PSD polynomials of degree $m = 6$. In this case from [6] we know that if a PSD polynomial has exactly ten zeros in $\mathbb{PR}^3$, than it cannot be SOS. Fixing the variable z generally reduced the amount of zeros and hence permits to find a lower order SOS polynomial with the same zeros.

The methods described above, together with the criterion (6), can be used to classify the states according to the degree of SOS polynomial detecting them. Let us define a family of convex subsets of $\mathcal{P}$:

$$S_m := \bigcap_{w \in \tilde{\Sigma}_m} \{P \in \mathcal{P}\,;\, \int d^2\alpha P\, w \geq 0\}, \tag{15}$$

where $\tilde{\Sigma}_m$ is the set of (inhomogeneous) SOS polynomials of degree m. Theorem (6) implies that $\Delta = \bigcap_k S_{2k}$. It is also clear that $\tilde{\Sigma}_2 \subset \tilde{\Sigma}_4 \subset \cdots$ and hence $S_2 \supset S_4 \supset \cdots$. We prove a stronger result.

Theorem.—For any even m there exist nonclassical states detected by some witness from $\tilde{\Sigma}_m$, and not by any witness from $\tilde{\Sigma}_{(m-2)}$, that is $S_2 \supsetneq S_4 \supsetneq \ldots$.

Proof.— Let us choose a generic $w \in \tilde{\Sigma}_m$. It has $(m + 1)(m + 2)/2$ terms, as it is a sum of polynomials of degree $\leq m$. From the variety $V(w) := \{(x, y)\,;\, w(x, y) = 0\}$ we pick n points $(x_1, y_1), \ldots, (x_n, y_n)$, $m(m + 1)/2 < n < (m + 1)(m + 2)/2$, such that they do not lie on any variety of the lower order $V(u)$, $u \in \tilde{\Sigma}_{(m-2)}$. We can find such points, as otherwise there would exist $u \in \tilde{\Sigma}_{(m-2)}$, such that $(x_1, y_1), \ldots, (x_n, y_n) \in V(u)$. However, with chosen n the latter condition leads to an overcomplete system of linear homogeneous equations for the coefficients of u, which generically possesses no solution. On the other hand, the same condition for $V(w)$ yields an underdetermined system possessing a nontrivial solution. Having such points we construct coherent states $|x_1 + iy_1\rangle, \ldots, |x_n + iy_n\rangle$ and a face $\mathcal{F}_n \in \Delta$ spanned by them. For any $\tilde{\varrho} \in \mathcal{F}_n$ we have then that $\mathrm{tr}\{\tilde{\varrho}{:}w(a, a^\dagger){:}\} = 0$, whereas $\mathrm{tr}\{\tilde{\varrho}{:}u(a, a^\dagger){:}\} > 0$ for all $u \in \tilde{\Sigma}_{(m-2)}$. Hence we can find such a convex combination ϱ of $\tilde{\varrho}$ and a projector onto some linear combination of $|x_1 + iy_1\rangle, \ldots, |x_n + iy_n\rangle$, such that $\mathrm{tr}\{\varrho{:}w(a, a^\dagger){:}\} < 0$, while for all $u \in \tilde{\Sigma}_{(m-2)}$, $\mathrm{tr}\{\varrho{:}u(a, a^\dagger){:}\} \geq 0$ (from the continuity). □

Summarizing, we have derived a family of classicality criteria of states of a quantum system, that require that P representation averages of positive functions are positive. For polynomial functions, we have related these criteria to Hilbert's 17th problem: we have proven the theorem that all "generic" nonclassical states (for which the P representation averages of polynomials exist), can be detected by SOS polynomials of sufficiently high degree; in this sense, non-SOS polynomials (whose existence was proven by Hilbert) are not necessary for classicality detection. We have also introduced the hierarchy of states implied by this theorem, and have introduced convex sets $S_2 \supsetneq S_4 \supsetneq \ldots$ of states detected by squares of polynomials of the 1st, 2nd, ..., order, and corresponding in this sense to decreasing degree of quantumness.

We stress that our results have important experimental consequences. Our polynomial nonclassicality witnesses can be easily measured, allowing thus for direct detection of quantumness and its degree for a given state. In this sense they are similar to entanglement witnesses that are nowadays commonly used for detection of entanglement [20]. If one wants to check if a given state ϱ is quantum, it is enough to measure normally ordered averages of squares of real polynomials of position q, and momentum p, or quadrature operators. In order to check the degree of quantumness (i.e., to check whether $\varrho \in S_{2k}$), one should determine normally ordered averages of squares of real polynomials of the order k. Note, that for a given k this requires measurements of finite number of averages only. For instance, for $k = 1$ (squeezing), one needs to measure $\langle q\rangle$, $\langle p\rangle$, $\langle{:}q^2{:}\rangle$, $\langle{:}p^2{:}\rangle$, and $\langle{:}qp + pq{:}\rangle$, and check if there exist A, B, C such that $\langle{:}(Aq + Bp + C)^2{:}\rangle < 0$. For general k, one needs, respectively, $k(2k + 3)$ measurements. Our results for the first time fully categorize states with respect to their degree of quantumness, and generalize the concepts of (higher order) squeezing or number squeezing as a signature of quantumness.

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