New Concept of Relativistic Invariance in Noncommutative Space-Time: Twisted Poincaré Symmetry and Its Implications

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We present a systematic framework for noncommutative (NC) quantum field theory (QFT) within the new concept of relativistic invariance based on the notion of twisted Poincaré symmetry, as proposed by Chaichian *et al.* [Phys. Lett. B **604**, 98 (2004)]. This allows us to formulate and investigate all fundamental issues of relativistic QFT and offers a firm frame for the classification of particles according to the representation theory of the twisted Poincaré symmetry and as a result for the NC versions of *CPT* and spin-statistics theorems, among others, discussed earlier in the literature. As a further application of this new concept of relativism we prove the NC analog of Haag's theorem.

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Introduction.—The idea that the space-time may be not a continuous manifold, but a quantized object, which can be expressed through the noncommutativity of coordinate operators, can be traced back to Heisenberg. Powerful arguments coming both from string theory in a background field [1] and from quantum mechanics and general relativity [2] indicate that, when gravitational effects come into play, at Planck scale, the space-time coordinates x^μ should indeed be replaced by the Hermitian operators $\hat{x}^\mu$, satisfying the commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is a *constant* antisymmetric matrix. Quantum field theories (QFTs) have been formulated on such noncommutative (NC) space-times and investigated perturbatively (for reviews, see [3]), using the Weyl-Moyal correspondence, which associates with a field operator $\phi(\hat{x})$ its Weyl symbol $\phi(x)$ defined on the commutative counterpart of the noncommutative space-time. As a consequence, the products of operators, $\phi(\hat{x})\psi(\hat{x})$, are replaced by the Moyal $\star$-products of Weyl symbols $\phi(x)\star\psi(x)$, defined as

$$\phi(x)\star\psi(x) = \phi(x)\exp\left[\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right]\psi(y)|_{x=y} \quad (2)$$

Such NC QFTs differ in two major aspects from their commutative counterparts: they are nonlocal and violate Lorentz invariance (but preserve translational symmetry). Though these features do not pose crucial problems for the perturbative approach, they are of essential importance when an axiomatic formulation is attempted, since relativistic (Poincaré) invariance and locality are two of the very axioms of the usual QFT. Moreover, the space-time symmetry is also necessary for the formulation of the other axioms, as they have to be expressed in terms of the invariants of the theory.

With a suitable choice of coordinates, such that $\theta^{12} \neq 0$ and $\theta^{03} \neq 0$ (the other components of the θ matrix being zero), it is easy to see that the largest subgroup of the Lorentz group which preserves the θ matrix is $SO(1,1)\times SO(2)$, formed by boosts in the 3rd space direction and rotations in the $(1,2)$ plane. From perturbative calculations, it is well-known that in the case of space-time noncommutativity, i.e., when some of $\theta^{0i} \neq 0$, $i = 1, 2, 3$, the resulting theory violates causality and unitarity [4–6]. Therefore, we consider only space-space noncommutativity with a choice of space coordinates such that only $\theta^{12} = -\theta^{21} \equiv \theta$ is not vanishing [in which case the Lorentz symmetry is broken down to the *residual* $O(1,1)\times SO(2)$ symmetry].

An axiomatic formulation of NC QFT, based on this residual space-time symmetry, was proposed in [7]. The resulting theory is effectively a $(1+1)$-dimensional QFT in the $(0,3)$-Minkowski plane, with instantaneous interactions in the $(1,2)$ Euclidean plane. The methods of usual (commutative) relativistic QFT are applied in the $(0,3)$-Minkowski plane, whereas the role of NC coordinates $\hat{x}^1$ and $\hat{x}^2$ is strongly suppressed. This enables us to investigate *some* general aspects which can be adapted to the residual space-time symmetry. However, this approach cannot be considered to be satisfactory: (i) There are no obvious reasons why the full relativistic symmetry should emerge in the commutative limit $\theta \to 0$, and (ii) since both $O(1,1)$ and $SO(2)$ are *Abelian* groups in four dimensions, they have only one-dimensional (i.e., scalar) unitary irreducible representations; therefore there is no natural way how to introduce the spin and the classification of particles.

Recently, in [8] a "hidden" symmetry of NC QFT was revealed, which lead to an alternative interpretation of the commutation relations (1). Using the notion of *twisted Poincaré* symmetry (with the same ten generators, i.e., four energy-momentum generators, P_μ, and six Lorentz generators, $M_{\mu\nu}$, and thus with exactly the same representations as the usual Poincaré algebra), the interpretation of (1) was extended from the Lie algebra framework to Hopf algebras: *If in the usual (commutative) case, relativistic* *invariance means symmetry under the Poincaré transformations, in the noncommutative case relativistic invariance means symmetry under twisted Poincaré transformations*. This symmetry enables us to discuss all the aspects of relativistic QFT, which are not accessible within the residual symmetry approach. For example, it justifies the attempt to prove the spin-statistics theorem in [5] in Lagrangian formulation and in [9] within the axiomatic approach.

The main aim of this Letter is to present a consistent frame for NC QFT, in particular, for its axiomatic approach, with the relativistic invariance realized in terms of the twisted Poincaré symmetry proposed previously and to apply the developed techniques to the investigation and proof of the Haag theorem [10], with the statement: If a field at a certain time is related to a free one by a unitary transformation, as is the case in the interaction picture, then the field is inevitably free. This theorem concerns deep mathematical subtleties of systems possessing an infinite number of degrees of freedom, as is the case of QFT. It is an ultimate question to understand its background and nature also in the framework of NC QFT. Since the latter theories are nonlocal, one might think that the analog of Haag's theorem may no more be valid due to the nature of NC field theories: while interacting fields are nonlocal, the free fields are local.

Twisted NC QFT.—The mathematical details of the abstract construction of the twisted Poincaré Hopf algebra in the context of NC QFT have been given in [8] (for basic notions on quantum groups, see [11]). At this point we emphasize only its main physical requirements and implications.

The twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ is a deformation of the enveloping $\mathcal{U}(\mathcal{P})$ of the usual Poincaré algebra $\mathcal{P}$. The deformation is achieved by a twist element, constructed with the generators P_μ of the Abelian algebra of translations [recall that the commutation relation (1) is translational invariant]. The twist does not affect the generators of the algebra, which remain the same as the generators of $\mathcal{U}(\mathcal{P})$, with the major consequence that the representations of $\mathcal{U}_t(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})$ are the same. What is essentially changed is the *coproduct*, i.e., the action of the generators in the tensor product of representations. For consistency, the twist also modifies the multiplication law in the algebra of the representation.

To emphasize the physical meaning of the twist deformation, let us take the concrete example of the algebra $\hat{\mathcal{A}}$ of operators in QFT Hilbert space $\mathcal{H}$ generated by products of fields $\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)$, where $x_1,\ldots,x_n$ are space-time points in the Minkowski space. We assume that $\mathcal{H}$ carries a unitary representation of Poincaré algebra $\mathcal{P}$ with the self-adjoint generators $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$. Furthermore, we assume that there is a unique $\mathcal{P}$-invariant vacuum state $|0\rangle \in \mathcal{H}$: $\hat{P}_\mu|0\rangle = \hat{M}_{\mu\nu}|0\rangle = 0$, and that $|0\rangle$ is a cyclic vector in Hilbert space, i.e., $\mathcal{H}$ is spanned by vectors $\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$, $x_1,\ldots,x_n \in M$.

The operators $\hat{P}_\mu$ and $\hat{M}_{\mu\nu}$ act on individual fields in the standard way:

$$[\hat{P}_\mu, \hat{\phi}(x)] = -i\partial_\mu\hat{\phi}(x) \equiv (\mathcal{P}_\mu\hat{\phi})(x),$$
$$[\hat{M}_{\mu\nu}, \hat{\phi}(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\hat{\phi}(x) \equiv (\mathcal{M}_{\mu\nu}\hat{\phi})(x). \quad (3)$$

So far, there is no difference between $\mathcal{U}_t(\mathcal{P})$ and $\mathcal{U}(\mathcal{P})$, the basic algebraic structure not being affected. However, when one extends the above action of the *Lorentz generators* via commutators to any product of fields, the differences become evident (the action of the translation generators, in the two cases, are still the same).

In the usual (commutative) case, the multiplication in the algebra $\hat{\mathcal{A}}$ of the representation of $\mathcal{U}(\mathcal{P})$ is the usual product of functions, $\hat{\phi}(x)\hat{\phi}(y)$. The Lorentz generator $\hat{M}_\omega = \frac{1}{2}\omega^{\mu\nu}\hat{M}_{\mu\nu}$, $\omega^{\mu\nu} = -\omega^{\nu\mu}$, acts on the product $\hat{\phi}(x)\hat{\phi}(y)$ as follows:

$$\begin{aligned}\mathcal{M}_\omega(\hat{\phi}(x)\hat{\phi}(y)) &\equiv [\hat{M}_\omega, \hat{\phi}(x)\hat{\phi}(y)] \\ &= [\hat{M}_\omega, \hat{\phi}(x)]\hat{\phi}(y) + \hat{\phi}(x)[\hat{M}_\omega, \hat{\phi}(y)] \\ &= (\mathcal{M}_\omega\hat{\phi})(x)\hat{\phi}(y) + \hat{\phi}(x)(\mathcal{M}_\omega\hat{\phi})(y).\end{aligned} \quad (4)$$

In the twisted (noncommutative) case, we arrive at the field operator algebra $\hat{\mathcal{A}}_\star$ carrying the representation of $\mathcal{U}_t(\mathcal{P})$. As shown explicitly in [8], the multiplication in $\hat{\mathcal{A}}_\star$ is affected by the twist, resulting in a $\star$-multiplication:

$$\hat{\phi}(x)\star\hat{\phi}(y) = \exp\left[\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}\right]\hat{\phi}(x)\hat{\phi}(y). \quad (5)$$

We stress that although (5) looks like a Moyal product, there has been *no* noncommutativity of coordinates used. The $\star$-product is *inherited* from the twist of the operator product of quantum fields.

In accord with the discussion of [8] [Eq. (10)], the Lorentz generator $\hat{M}_\omega$ defined above acts practically on the $\star$-product of fields, as follows (see also [12]):

$$\begin{aligned}\mathcal{M}^t_\omega(\hat{\phi}(x)\star\hat{\phi}(y)) = {}& (\mathcal{M}_\omega\hat{\phi})(x)\star\hat{\phi}(y) + \hat{\phi}(x) \\ &\star(\mathcal{M}_\omega\hat{\phi})(y) - \tfrac{1}{2}\theta^{\rho\sigma}\omega^\nu_\rho(\mathcal{P}_\nu\hat{\phi})(x) \\ &\star(\mathcal{P}_\sigma\hat{\phi})(y) - \tfrac{1}{2}\theta^{\rho\sigma}\omega^\nu_\sigma(\mathcal{P}_\rho\hat{\phi})(x) \\ &\star(\mathcal{P}_\nu\hat{\phi})(y)\end{aligned} \quad (6)$$

By this apparently complicated action ("twisted coproduct"), the theory of twist deformations [11] ensures the *twisted Poincaré covariance* of the $\star$-product of fields:

$$\begin{aligned}\hat{\phi}(x)\star\hat{\phi}(y) &= \exp\left[\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}\right]\hat{\phi}(x)\hat{\phi}(y) \\ &\mapsto \exp\left[\frac{i}{2}\theta^{\mu\nu}\partial_{x'^\mu}\partial_{y'^\nu}\right]\hat{\phi}(x')\hat{\phi}(y') \\ &= \hat{\phi}(x')\star\hat{\phi}(y'),\end{aligned} \quad (7)$$

where $x' = \Lambda x + a$ and $y' = \Lambda y + a$ are the accordingly transformed space-time points.

It is therefore clear that, for NC QFT, the axiom of relativistic invariance has to be replaced by the requirement of twisted Poincaré symmetry.

Spectral condition.—We recall that in the twisted Poincaré context P^2 retains its role as invariant operator. Taking into account also that we consider *only* theories with space-space noncommutativity, $\theta_{0i} = 0$, which are well defined (unitary, causal, and can be obtained as low-energy limit from string theory), we postulate that P_μ has the usual spectrum $V_+ = \{p^\mu; p^0 \geq |\vec{p}|\}$; i.e., physical states have momentum in the forward light cone and form a complete set. For a free field theory, the Hilbert space in the NC case coincides with the one of usual QFT. This consistency gives a justification for the spectral postulate, at least perturbatively, also in the case of interacting fields.

Locality condition.—The locality condition is an independent axiom. The covariance property (7) *dictates* the relativistic form of locality condition among $\star$-products of fields. We postulate it in the simplest form:

$$[\hat{\phi}(x), \hat{\phi}(y)]_\star = 0, \quad \text{for } (x-y)^2 = (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 < 0. \tag{8}$$

Note that in [13], within the NC context, a weaker locality condition was proposed in the form

$$[\hat{\phi}(x), \hat{\phi}(y)]_\star = 0, \quad \text{for } (x-y)^2 < -l^2, \tag{9}$$

with $l^2 \approx |\theta|$. In [14] it was proven that in a field theory satisfying *translational invariance* and *spectral condition* $p \in V_+$, the locality condition (9) implies (8). In our case, these requirements are indeed satisfied, thus supporting the use of the form (8) for the locality axiom.

Wightman functions.—The NC Wightman functions are defined as the vacuum expectation values of multiple $\star$-products of fields [9]:

$$\begin{aligned} W_\star(x_1, \ldots, x_n) &\equiv \langle 0|\hat{\phi}(x_1) \star \cdots \star \hat{\phi}(x_n)|0\rangle \\ &= \exp\left[\frac{i}{2}\theta^{\mu\nu}\sum_{a<b}\partial_{x_a^\mu}\partial_{x_b^\nu}\right] W(x_1, \ldots, x_n), \end{aligned} \tag{10}$$

where $W(x_1, \ldots, x_n) = \langle 0|\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|0\rangle$. Because of translational invariance, the θ-dependent exponent in (10) can be omitted for the 2-point NC Wightman function: $W_\star(x, y) = W(x, y) = \mathcal{W}(x - y)$. However, it cannot be omitted for the higher NC Wightman functions.

The NC Wightman functions defined above satisfy the twisted Poincaré covariance condition and the twisted Poincaré locality condition:

$$\begin{aligned} &W_\star(x_1, \ldots, x_n) \mapsto W_\star(\Lambda x_1 + a, \ldots, \Lambda x_n + a), \\ &W_\star(\ldots, x, y, \ldots) = W_\star(\ldots, y, x, \ldots), \quad \text{for } (x-y)^2 < 0, \end{aligned} \tag{11}$$

together with the usual positivity axiom (the positivity of the norm in $\mathcal{H}$).

This enables us to formulate and discuss various fundamental NC QFT theorems along the lines similar to the standard ones.

Haag's theorem.—As an interesting application of the twisted Poincaré formalism we prove Haag's theorem in the case of NC QFT with space-space noncommutativity, $\theta_{0i} = 0$. The theorem was originally formulated in [10]; its variations and extensions were presented in [15] (for a detailed discussion, see [16]). Below we briefly discuss a sequence of theorems leading, in the twisted Poincaré context, to Haag's theorem. The first theorem we need can be formulated as follows:

Theorem 1: *Let the 2-point function* $\langle 0|\hat{\phi}(x) \star \hat{\phi}(y)|0\rangle$ *of an arbitrary scalar field* $\hat{\phi}(x)$ *coincide with the 2-point function of a free field of mass* $m > 0$, $-i\Delta(x - y; m)$, *satisfying the Klein-Gordon equation, so that*

$$(\partial_x^2 + m^2)\langle 0|\hat{\phi}(x) \star \hat{\phi}(y)|0\rangle = -i(\partial_x^2 + m^2)\Delta(x - y; m) = 0. \tag{12}$$

Then the interaction current vanishes: $\hat{J}(x) \equiv (\partial_x^2 + m^2)\hat{\phi}(x) = 0$, i.e., *the theory is free*.

Proof: The first part of the proof is identical to the usual one [15]. Considering an arbitrary test function $F(x)$ we prove that the state $|\psi\rangle \equiv \int dx F(x)\hat{J}(x)|0\rangle = \int dx F(x) \times (\partial_x^2 + m^2)\hat{\phi}(x)|0\rangle$ vanishes. Omitting the $\star$-product in (12) we obtain

$$\begin{aligned} \langle\psi|\psi\rangle &= \int dx dy \bar{F}(x) F(y) (\partial_x^2 + m^2)(\partial_y^2 + m^2) \\ &\quad \times \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \\ &= 0. \end{aligned}$$

Since $F(x)$ is arbitrary, the current $\hat{J}(x)$ annihilates the vacuum state: $\hat{J}(x)|0\rangle = 0$.

The nontrivial part of the theorem is based on the existence of NC analogs of Jost points for the Wightman functions $W_\star(x_1, \ldots, x_{n-1}, x_n) = \langle 0|\hat{\phi}(x_1) \star \cdots \star \hat{\phi}(x_{n-1}) \star \hat{\phi}(x_n)|0\rangle$ which are boundary values of a holomorphic function $W_\star(z_1, \ldots, z_{n-1}, z_n)$ of complex variables $z_1, \ldots, z_{n-1}, z_n$ in a proper domain (the extended tube) containing real Jost points $(r_1, \ldots, r_{n-1}, r_n)$, satisfying $(r_k - r_l)^2 < 0$, for all $k > l$. Because of the locality condition (11), we can permute the fields evaluated at Jost points: $W_\star(r_1, \ldots, r_{n-1}, r_n) = W_\star(r_1, \ldots, r_n, \ldots, r_{n-1})$. Performing now in this equation the analytical continuation back to $x_1 \ldots, x_{n-1}, x_n$, and acting by $(\partial_{x_n}^2 + m^2)$, we obtain

$$\begin{aligned} &\langle 0|\hat{\phi}(x_1) \star \cdots \star \hat{\phi}(x_{n-1}) \star \hat{J}(x_n)|0\rangle \\ &\quad = \langle 0|\hat{\phi}(x_1) \star \cdots \star \hat{J}(x_n) \star \cdots \star \hat{\phi}(x_{n-1})|0\rangle. \end{aligned} \tag{13}$$

Since $\hat{J}(x)|0\rangle = 0$, the left-hand side of (13) is zero. The right-hand side represents, due to the completeness axiom, an arbitrary matrix element of the interaction current. Consequently, $\hat{J}(x) = 0$; i.e., the theory is *free*.

To prove Haag's theorem we need a simple theorem dealing with fixed time fields. Its formulation and proof is similar to the usual one (see [16]).

Theorem 2: *Let $\hat{\phi}_1(\vec{x}, t)$ and $\hat{\phi}_2(\vec{x}, t)$ be two irreducible scalar fields at a fixed time t defined, respectively, in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in which there are two continuous unitary representations of the Euclidean group E(3):*

$$U_j(\vec{a}, R)\hat{\phi}_j(\vec{x}, t)U_j^\dagger(\vec{a}, R) = \hat{\phi}_j(R\vec{x} + \vec{a}, t), \qquad j = 1, 2.$$

We assume that (i) the representations possess unique invariant vacuum states $|0\rangle_j$: $U_j(\vec{a}, R)|0\rangle_j = |0\rangle_j$, $j = 1$, 2, and (ii) there exists a unitary operator V: $\mathcal{H}_1 \to \mathcal{H}_2$ such that, at time t, $\hat{\phi}_2(\vec{x}, t) = V\hat{\phi}_1(\vec{x}, t)V^\dagger$. Then $U_2(\vec{a}, R) = VU_1(\vec{a}, R)V^\dagger$, $|0\rangle_2 = e^{i\alpha}V|0\rangle_1$, $\alpha \in \mathbf{R}^1$.

Note that the group $E(3)$ is a subgroup of the Poincaré group generated by rotations and translations in $\mathbf{R}^3$. This induces the representation of the twisted Euclidean Hopf algebra in the operator algebra generated by field operators at the fixed time t. It is a Hopf subalgebra of $\hat{\mathcal{A}}_\star$; this is the place where the conditions $\theta^{0i} = 0$, $i = 1, 2, 3$, become essential.

Corollary: The equal-time vacuum expectation values of both theories coincide: ${}_1\langle 0|\hat{\phi}_1(\vec{x}_1, t) \star \cdots \star \hat{\phi}_1(\vec{x}_n, t)|0\rangle_1 = {}_2\langle 0|\hat{\phi}_2(\vec{x}_1, t) \star \cdots \star \hat{\phi}_2(\vec{x}_n, t)|0\rangle_2$.

Theorem 1 and the Corollary provide us with the proof of Haag's theorem:

Theorem 3 (Haag): *Suppose $\hat{\phi}_1(x)$ is a free Hermitian scalar field of mass $m > 0$, and $\hat{\phi}_2(x)$ is a scalar field satisfying the NC QFT axioms given above. Suppose further that the fields $\hat{\phi}_j(x)$, $(\partial_t\hat{\phi})_j(x)$, $j = 1, 2$, satisfy the hypothesis of Theorem 2. Then $\hat{\phi}_2(x)$ is a free field of mass m.*

Proof: Any two spacelike separated points x and y can be brought by a Lorentz transformation to equal-time plane: $x = (\vec{x}, t)$ and $y = (\vec{y}, t)$. By the Corollary we have ${}_2\langle 0|\hat{\phi}_2(\vec{x}, t) \star \hat{\phi}_2(\vec{y}, t)|0\rangle_2 = -\mathrm{i}\Delta(\vec{x} - \vec{y}, 0; m)$. Using the standard analytic continuation argument and the twisted relativistic covariance of $\hat{\phi}_2(x) \star \hat{\phi}_2(y)$ we obtain ${}_2\langle 0|\hat{\phi}_2(x) \star \hat{\phi}_2(y)|0\rangle_2 = -i\Delta(x - y; m)$. The Haag's theorem is then a direct consequence of Theorem 1.

Concluding remarks.—The approach presented above is based on the twisted Poincaré symmetry which states that, while symmetry under the usual Lorentz transformations guarantees the relativistic invariance of a theory, in the NC QFT the concept of relativistic invariance, however, should be replaced by the requirement of invariance of the theory under the twisted Poincaré transformations. The latter allows us to formulate and discuss all fundamental issues of relativistic QFT within a NC context. Because of the content of its representations, the twisted Poincaré algebra offers a firm framework for the proofs of the NC version of *CPT* and the spin-statistics theorems [9], among other results obtained in the literature so far, and justifies, in particular, the results obtained in [5] within Lagrangian formulation. The Haag theorem also belongs to that class of exact (not perturbative) results.

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