Resonant Nucleation

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(Received 28 August 2004; revised manuscript received 28 January 2005; published 20 April 2005)

We investigate the role played by fast quenching on the decay of metastable (or false vacuum) states. Instead of the exponentially slow decay rate per unit volume, $\Gamma_{\rm HN} \sim \exp[-E_b/k_BT]$ (E_b is the free energy of the critical bubble), predicted by homogeneous nucleation theory, we show that under fast enough quenching the decay rate is a power law $\Gamma_{\rm RN} \sim [E_b/k_BT]^{-B}$, where B is weakly sensitive to the temperature. For a range of parameters, large-amplitude oscillations about the metastable state trigger the resonant emergence of coherent subcritical configurations. Decay mechanisms for different E_b are proposed and illustrated in a $(2+1)$-dimensional scalar field model.

DOI: 10.1103/PhysRevLett.94.151601 PACS numbers: 11.10.Lm, 05.45.Xt, 64.60.Qb, 98.80.Cq

Introduction.—Few topics in physics have the range of applicability of first order (or discontinuous) phase transitions. From materials science to particle physics and cosmology, the fact that a large number of physical systems can be described as having two phases separated by an energy barrier has been an active topic of research for decades [1–3]. Much of the theoretical work in this field is derived in one way or another from the theory of homogeneous nucleation (HN) [1,4]. HN assumes that the system is initially localized in a spatially homogeneous metastable state, that is, that only small fluctuations about the local equilibrium state ϕ_m exist: one computes the partition function $Z = -T \ln\mathcal{F}$ summing only over quadratic fluctuations about ϕ_m. $\mathcal{F}$ is the free energy, given by the path integral $\int \mathcal{D}\phi \exp[-E[\phi]/T]$, where $E[\phi]$ is the free-energy functional of configuration ϕ. [$k_B = c = \hbar = 1$ throughout.] In relativistic field theory, false vacuum decay has been examined both at zero [5] and finite temperature [6] by a large number of authors [7]. In general, the HN approximation is adopted from the start. At finite temperatures, one uses the well-known exponential decay rate per unit volume, $\Gamma_{\rm HN} \simeq T^{(d+1)} \exp[-E_b/T]$, where E_b is the free-energy barrier for the decay, or the energy of the critical bubble or bounce, $\phi_b(r)$, the solution to the equation $\phi'' + \frac{d-1}{r}\phi' = \frac{\partial V[\phi]}{\partial\phi}$, with appropriate boundary conditions. $\phi' \equiv \partial\phi/\partial r$ and $V[\phi]$ is the effective potential that sums over thermal and quantum contributions when applicable. At $T = 0$, the prefactor $T^{(d+1)}$ is roughly approximated by $M^{(d+1)}$, the relevant mass scale, while E_b/T is substituted by $S_E[\phi_b]$, the $(d+1)$-dimensional Euclidean action of the bounce configuration.

In the present work we examine what happens if one relaxes the HN approximation that the initial state is well-localized about equilibrium. We subject the system to an instantaneous quench, equivalent to a sudden change of potential from a single to an asymmetric double well. This should be contrasted with the work of Ref. [8] which studies quenches in models with spinodal decomposition dynamics. Although in this first study we will only consider instantaneous quenches, we expect our results to carry on at least partially to slower quenches, so long as the quenching rate $\tau_{\rm quench}$ is faster than the relaxation rate of the field's zero mode, τ_0. With the longest wavelength being the slowest to equilibrate, as $\tau_{\rm quench} \rightarrow \tau_0$, the system will remain in equilibrium. For appropriate choices of parameters, the rapid quench will induce large-amplitude oscillations of the field's zero mode [9]. Because of the nonlinear potential, energy will be transferred from the zero mode to higher k modes. As observed in reference [9], this transfer of energy results in the synchronous emergence of oscillonlike configurations [10]. For small enough double-well asymmetry, these localized field configurations act as precursors for the nucleation of a critical bubble, greatly reducing the decay time scale. In the simulations we examined, the critical bubble emerges as two or more subcritical oscillons coalesce, or, for larger asymmetries, as a single oscillon becomes critically unstable to growth. If the asymmetry is too large, the field crosses directly to the global minimum.

The Model.—Consider a $(2+1)$-dimensional real scalar field $\phi(\mathbf{x}, t)$ evolving under the influence of a potential $V(\phi)$. The continuum Hamiltonian is conserved and the total energy of a given field configuration $\phi(\mathbf{x}, t)$ is

$$H[\phi] = \int d^2x\left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right], \quad (1)$$

where $V(\phi) = \frac{m^2}{2}\phi^2 - \frac{\alpha}{3}\phi^3 + \frac{\lambda}{8}\phi^4$ is the potential energy density. The parameters m, α, and λ are positive definite and temperature independent. It is helpful to introduce the dimensionless variables $\phi' = \phi\sqrt{\lambda}/m$, $x' = xm$, $t' = tm$, and $\alpha' = \alpha/(m\sqrt{\lambda})$ (we will henceforth drop the primes). Prior to the quech, $\alpha = 0$ and the potential is an anharmonic single well symmetric about $\phi = 0$. The field is in thermal equilibrium with a temperature T. At the temperatures considered, the fluctuations of the field are well approximated by a Gaussian distribution, with $\langle \phi^2 \rangle = aT$ ($a = 0.51$ and can be computed numerically [11]). As such, within the context of the Hartree approximation [12], the momentum and field modes in k-space can be obtained from a harmonic effective potential, and satisfy $\langle |\bar{\pi}(k)|^2 \rangle = T$ and $\langle |\bar{\phi}(\mathbf{k})|^2 \rangle = T/(k^2 + m_H^2)$, respectively. The Hartree mass $m_H^2 = 1 + \frac{3}{2}\langle \phi^2 \rangle$ depends on the magnitude of the fluctuations (and thus T). We will refer to a particular system by its initial temperature. Results are ensemble averages over 100 simulations.

If $\alpha \neq 0$, the Z_2 symmetry is explicitly broken. When $\alpha = 1.5 \equiv \alpha_c$, the potential is a symmetric double well, with two degenerate minima. The quench is implemented at $t = 0$ by setting $\alpha > \alpha_c$, whereby the potential is asymmetric about the barrier separating the two minima. The Hartree approximation gives an accurate description of the *early* evolution of the area-averaged field $\phi_{\text{ave}}(t)$ and its fluctuations while their distribution remains Gaussian and the dynamics are governed by an effective potential,

$$V_{\text{eff}}(\phi_{\text{ave}}, m_{\text{H}}^2) = \left[1 - m_{\text{H}}^2(t)\right]\phi_{\text{ave}} + \frac{1}{2} m_{\text{H}}^2(t)\phi_{\text{ave}}^2 - \frac{\alpha}{3}\phi_{\text{ave}}^3 + \frac{1}{8}\phi_{\text{ave}}^4. \tag{2}$$

The quench shifts the local minimum from $\phi_{\text{ave}} = 0$ to a positive value, but also introduces a new global minumum to the system. [See inset in Fig. 2.]

Nucleation of Oscillons.—The quench induces oscillations in ϕ_{ave} about the new local minimum, which eventually dampen due to nonlinear scattering with higher k modes. At early times small fluctuations satisfy a Mathieu equation in k space

$$\delta\ddot{\phi} = -[k^2 + V''_{\text{eff}}[\phi_{\text{ave}}(t)]]\delta\phi, \tag{3}$$

and, depending on the wave number and parametric oscillations of $\phi_{\text{ave}}(t)$, can undergo exponential amplification ($\sim e^{\eta t}$). In Fig. 1 we show the lines of constant amplification rate for different wave numbers and temperatures when $\alpha = \alpha_c$ and V_{eff} is defined by the initial thermal distribution. At low temperatures $T \lesssim 0.13$, no modes are ever amplified. As the temperature is increased, so is the amplitude and period of oscillation in ϕ_{ave}, gradually causing the band $0 < k < 0.48$ to resonate and grow. A full description of the coupled dynamics of $\phi_{\text{ave}}(t)$ and $\delta\phi(\mathbf{x}, t)$ when $\alpha = \alpha_c$ is given in Ref. [11]. For $T \gtrsim 0.13$, large-amplitude fluctuations about the zero mode probe into unstable regions where $V''_{\text{eff}} < 0$. Note that this is very distinct from spinodal decomposition, where competing domains of the metastable and stable phases coarsen [1]: for the values of T and α considered, ϕ_{ave} oscillates about the metastable minimum until a critical bubble of the stable phase grows to complete the transition.

These two processes result in the synchronous emergence of oscillonlike configurations (Fig. 3 in Ref. [9]), long-lived time-dependent localized field configurations which are well described by Gaussian profiles, $\phi_{\text{osc}}(t, r) \simeq \phi_a(t)\exp[-r^2/R^2]$ [10]. To strengthen our argument, note that within this Gaussian ansatz, an oscillon is comprised by modes within the band $0 < k \leq 2/R$. One of us has recently shown that, in d dimensions and for a potential V, the radius of an oscillon satisfies $R^2 \geq d/[\frac{1}{2}(2^{3/2}/3)^d \times (V''')^2/V^{IV} - V'']$ [13]. For the potential of Eq. (2) and $d = 2$, we obtain that the related band of wave numbers is $0 < k \lesssim 0.66$. Referring to Fig. 1, the reader can verify that these are also approximately the modes excited by parametric resonance.

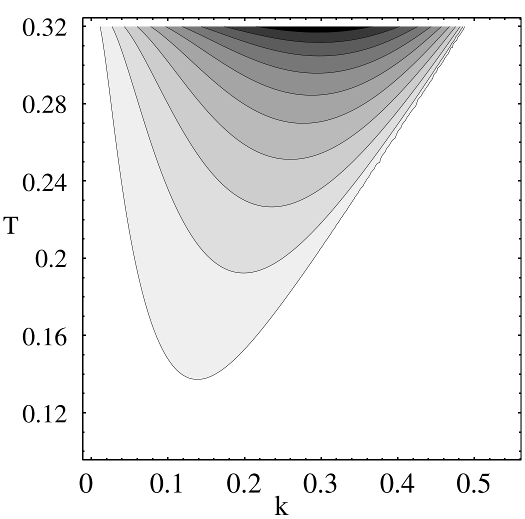

FIG. 1. Lines of constant amplification rate η for small-amplitude modes at various temperatures, beginning with $\eta_{\min} = 2.8 \times 10^{-2}$ for the bottom-most contour and increasing in increments of $\Delta\eta = 1.3 \times 10^{-2}$.

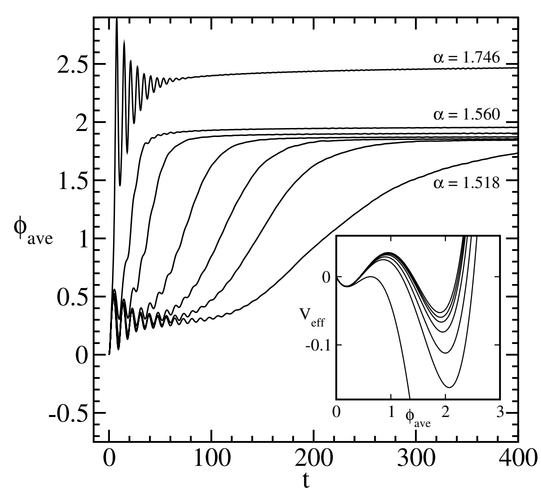

FIG. 2. The evolution of the order parameter $\phi_{\text{ave}}(t)$ at $T = 0.22$ for several values of the asymmetry. From left to right, $\alpha = 1.746, 1.56, 1.542, 1.53, 1.524, 1.521, 1.518$. The inset shows V_{eff} for the same values.

Resonant Nucleation.—Having established that oscillons emerge after the quench, we can examine their role as precursors of metastable decay. Unless the potential has a large asymmetry, oscillons are typically subcritical fluctuations; as will be discussed below, a critical nucleus may appear only due to the coalescence of two or more oscillons. It should be clear, however, that their appearance renders the homogeneity assumption of HN theory inapplicable: the metastable background is far from homogeneous and the critical energy barrier must be renormalized [14]. In other words, *a rapid quench or cooling leads to departures from the usual HN assumptions.* As we describe next, the decay rate of the quenched system may be much faster than what is predicted by HN theory.

In Fig. 2 we show the evolution of the order parameter $\phi_{\rm ave}(t)$ as a function of time for several values of asymmetry, $1.518 \leq \alpha \leq 1.746$, for $T = 0.22$. Not surprisingly, as $\alpha \rightarrow \alpha_c = 1.5$, the field remains longer in the metastable state, since the nucleation energy barrier $E_b \rightarrow \infty$ at α_c. However, a quick glance at the time axis shows the fast decay time scale, of order 10^{1-2}. For comparison, for $1.518 \leq \alpha \leq 1.56$, homogeneous nucleation would predict nucleation time scales of order $\sim 10^{28} \geq \tau_{\rm HN} \sim \exp[E_b/T] \geq 10^{12}$ (in dimensionless units). [The related nucleation barriers with the effective potential are $E_b(\alpha = 1.518) = 14.10$ and $E_b(\alpha = 1.56) = 5.74$.] While for smaller asymmetries $\phi_{\rm ave}(t)$ displays similar oscillatory behavior to the symmetric double well case before transitioning to the global minimum, as α is increased the number of oscillations decreases. For large asymmetries, $\alpha \geq 1.746$, the entire field crosses over to the global minimum without any nucleation event, resulting in oscillations about the global minimum. [See inset in Fig. 2.]

In Fig. 3 we show the ensemble-averaged nucleation time scales for resonant nucleation, $\tau_{\rm RN}$, as a function of the nucleation barrier [computed with Eq. (2)], E_b/T, for the temperatures $T = 0.18$, 0.20, and 0.22. [For temperatures above $T = 0.26$ no barrier exists.] The nucleation time was measured when $\phi_{\rm ave}$ crosses the maximum of $V_{\rm eff}$. The best fit is a power law, $\tau_{RN} \propto (E_b/T)^B$, with $B = 3.762 \pm 0.016$ for $T = 0.18$, $B = 3.074 \pm 0.015$ for $T = 0.20$, and $B = 2.637 \pm 0.018$ for $T = 0.22$. This simple power law holds for the same range of temperatures where we have observed the synchronous emergence of oscillons. It is not surprising that the exponent B increases with decreasing T, since the synchronous emergence of oscillons becomes less pronounced and eventually vanishes. At low T we should expect a smooth transition into the exponential time-scales of HN. We next present a possible mechanism by which the transition completes for different nucleation barriers.

First, for $\alpha \rightarrow \alpha_c$, the radius of the nucleation bubble diverges, $R_b \rightarrow \infty$. When fast quenching induces large-amplitude fluctuations of the field's zero mode, the system does not approach the global minimum through a random search in configuration space as is the case in HN. Instead, we argue that oscillons will induce the nucleation of a

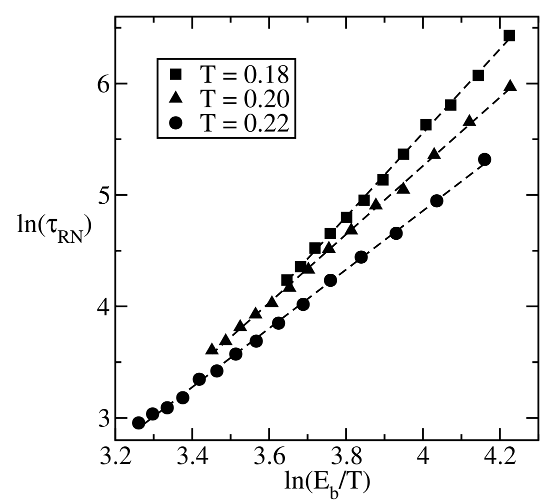

FIG. 3. Decay time-scale $\tau_{\rm RN}$ as a function of critical nucleation effective free-energy barrier E_b/T at $T = 0.18$, 0.20, and 0.22. The best fits (dashed lines) are power-laws with exponents $B \simeq 3.762$, 3.074, and 2.637, respectively.

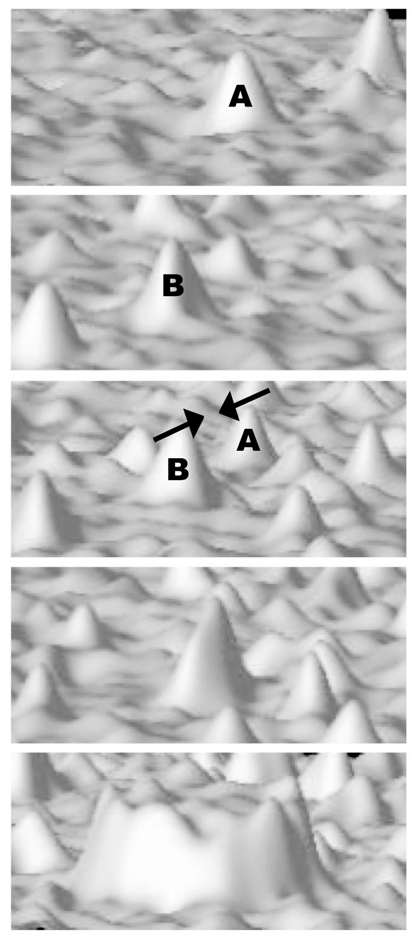

FIG. 4. Two oscillons coalesce to form a critical bubble. First two frames from top show oscillons A and B. Third and fourth frames show A and B coalescing into a critical bubble. Final frame shows growth of bubble expanding into metastable state.

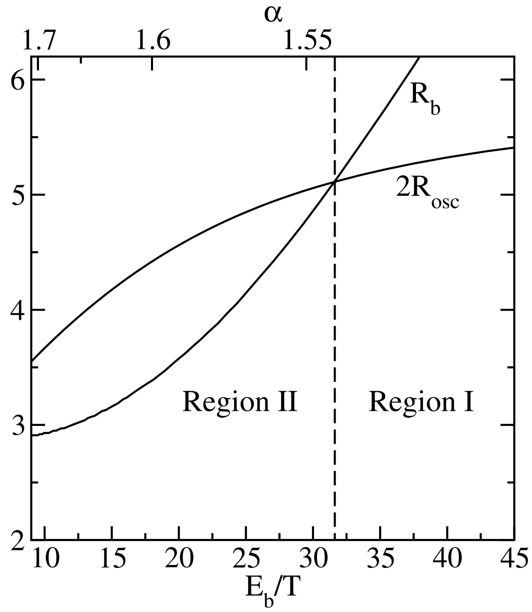

FIG. 5. Radius of critical bubble (R_b) and twice the minimum oscillon radius ($2R_{\rm osc}$) as a function of its energy barrier and related values of α at $T = 0.22$. For $\alpha \gtrsim 1.547$ one cannot easily distinguish between an oscillon and a critical bubble.

critical fluctuation. The way in which this happens depends on the magnitude of the nucleation barrier: for nearly degenerate potentials, $\alpha_c < \alpha \lesssim \alpha_{\rm I}$, the critical nucleus has a much larger radius than a typical oscillon; it will appear as two or more oscillons coalesce. We call this Region I, defined for $R_b \geq 2R_{\rm osc}$, where $R_{\rm osc}$ is the minimum oscillon radius computed from Ref. [13]. Figure 4 illustrates this mechanism. Two oscillons, labeled A and B, join to become a critical nucleus. They diffuse through the lattice and form bound states, somewhat as in kink-antikink breathers in 1D field theory [15,16].

As α is increased further, the radius of the critical nucleus decreases, approaching that of an oscillon. In this case, a single oscillon grows unstable to become the critical nucleus and promote the fast decay of the metastable state: there is no coalescence. We call this Region II, $\alpha_{\rm I} < \alpha \lesssim \alpha_{\rm II}$, $R_b < 2R_{\rm osc}$. This explains the small number of oscillations on $\phi_{\rm ave}(t)$ as α is increased [cf. Figure 2]. To corroborate our argument, in Fig. 5 we contrast the critical nucleation radius with that of oscillons as obtained in Ref. [13], for different values of effective energy barrier and related values of α at $T = 0.22$. The critical nucleus radius R_b is equal to $2R_{\rm osc}$ for $\alpha = 1.547$. This defines the boundary between Regions I and II: for $\alpha \gtrsim \alpha_{\rm I}$ a single oscillon may grow into a critical bubble. Finally, for $\alpha \gtrsim \alpha_{\rm II} = 1.746$ the field crosses over to the global minimum without any nucleation event.

How will the efficiency of the mechanism decrease as $\tau_{\rm quench} \to \tau_0$? What happens when the quench is induced by cooling as, for example, in the early Universe? Does the power law behavior obtained here still hold for $d = 3$? We intend to address these and related questions in the near future.

M. G. was partially supported by NSF grant no. PHYS-0099543. We would like to thank Jim Krumhansl, friend and mentor, for his insightful remarks.

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