Direct Processes in Chaotic Microwave Cavities in the Presence of Absorption

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We quantify the presence of direct processes in the S matrix of chaotic microwave cavities with absorption in the one-channel case. To this end the full distribution $P_S(S)$ of the S matrix, i.e., $S = \sqrt{R}e^{i\theta}$, is studied in cavities with time-reversal symmetry for different antenna coupling strengths T_a or direct processes. The experimental results are compared with random-matrix calculations and with numerical simulations including absorption. The theoretical result is a generalization of the Poisson kernel. The experimental and the numerical distributions are in excellent agreement with theoretical predictions for all cases.

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Random-matrix theory has been successfully applied to many different scattering systems in several branches of physics ranging from quantum mechanics and mesoscopic physics to sound and microwave systems [1–3]. The only precondition is the existence of chaotic ray dynamics of the system. Nevertheless, the connection between the statistical properties of scattering and the underlying chaos is not straightforward since scattering may involve two time scales, namely, a prompt and a delayed response. Prompt or direct processes are those in which the waves pass through the interaction region without a significant delay. In the equilibrated or delayed processes the waves suffer several reflections inside the interaction region. The delayed processes are usually studied with techniques of random-matrix theory, whereas the direct processes are described in terms of the average of the S matrix.

Starting from the pioneering work of López, Mello, and Seligman [4], there are several theoretical works addressing the statistical distributions of the S matrix with imperfect coupling or direct processes in the jargon of nuclear physics [5–10]. This distribution is known in literature as the Poisson kernel. In the one-channel case with no absorption it is possible to parametrize the S matrix as $S(E) = e^{i\theta(E)}$, and the Poisson kernel reads

$$p(\theta) = \frac{1}{2\pi}\frac{1 - |\langle S\rangle|^2}{|S - \langle S\rangle|^2}, \tag{1}$$

where $\langle S\rangle$ is the ensemble (or energy) average of $S(E)$. For $\langle S\rangle = 0$ the distribution of the phase $\theta(E)$ is uniform between 0 and 2π, i.e., $S(E)$ is uniformly distributed on the unitary circle, in agreement with the circular ensembles of random-matrix theory. Equation (1) means that the S-matrix distribution of a system including direct processes is fixed by the average $\langle S\rangle$ exclusively. In this Letter, we present measurements which provide clear evidence of direct processes in chaotic scattering.

Following Brouwer [8], the Poisson kernel can be interpreted as follows: For a chaotic system with an attached waveguide with ideal coupling, the distribution of the S matrix is uniform. If the coupling in the waveguide becomes nonideal, then the new S matrix is distributed according to the Poisson kernel. Thus any deviation from the uniform distribution (random-matrix result) yields information on the fraction of waves scattered without a significant delay.

A direct comparison between the Poisson kernel [Eq. (1)] and experimental results is usually not possible due to losses or absorption. Therefore it is important to take them into account. When absorption is present, S is a subunitary matrix. For the one-channel case the S matrix can be parametrized as

$$S = \sqrt{R}e^{i\theta}, \tag{2}$$

where R is the reflection coefficient. The coupling between the scattering channels and the interior region can be quantified by the transmission coefficient T_a of a barrier describing the direct processes:

$$T_a = 1 - |\langle S\rangle|^2. \tag{3}$$

The subindex a will be used to denote the antenna coupling. For perfect antenna coupling ($T_a = 1$) there are some exact results. The phase θ is uniformly distributed between 0 and 2π as before. The distribution $P_{R,0}(R)$ of R is known in the cases of strong ($\gamma \gg 1$) [11] and weak ($\gamma \ll 1$) absorption [12] for systems with and without time-reversal symmetry ($\beta = 1, 2$). Throughout this Letter the subindex "0" refers to perfect coupling. In case of systems without time-reversal symmetry and a single perfectly coupled channel, the distribution $P_{R,0}(R)$ was calculated for any absorption strength γ by Beenakker and Brouwer [12], as well as for two perfectly coupled channels in the presence of time-reversal symmetry. For systems with imperfect coupling and absorption, the distributions of the proper time delays and reflection eigenvalues are known only for $\beta = 2$ [13]. More recently, a general distribution of the reflection eigenvalues has been obtained for a large number of propagating channels, independent of time-reversal symmetry [14]. Other quantities, such as probabilities of no return, distributions of Wigner time delay have also been obtained for systems with absorption [15,16]. From the experimental point of view the effect of the absorption has been studied on the S-matrix correlation function [17,18], the reflection coefficient [17,19], the cross-correlation functions of the S matrix [20], the transmission coefficient [21], and very recently on resonance widths [22] and on the impedance matrix [23].

Here the distribution $P_\theta(\theta)$ of the phase of the subunitary S matrix given in Eq. (2) will be obtained. This corresponds to a generalization of Poisson's kernel, Eq. (1), including absorption. Also the full experimental distribution $P_S(S)$ of the S matrix in the presence of both absorption and direct processes for the one-channel case is given. Both distribution are related through $P_\theta(\theta) = \int_0^1 P_S(S) dR$.

To derive the distribution $P_S(S)$ of the S matrix, we consider the systems of Fig. 1. In Fig. 1(a) a system with absorption strength γ and perfect coupling ($T_a = 1$) is shown. Let us denote the S matrix describing the scattering in that case by $S_0 = \sqrt{R_0} e^{i\theta_0}$. We assume that the distribution $P_{S,0}(S_0)$ of S_0 is known. Actually, it is sufficient to know the distribution of the reflection coefficient $P_{R,0}(R_0)$ under the assumption that R_0 and θ_0 are still uncorrelated when absorption is present. As we mention above, the phase is uniformly distributed in the case of perfect coupling and no absorption, i.e.,

$$P_{S,0}(S_0) = \frac{1}{2\pi} P_{R,0}(R_0). \tag{4}$$

Just as for systems without absorption, the S matrix of the system with direct processes and absorption given by Eq. (2) can be written in terms of the S matrix of the system with absorption but without direct processes, S_0, and the S matrix of the barrier S_a that describes the nonideal coupling of the antenna [see Fig. 1(b)]. As a model [8,9], we take the S matrix of the barrier to be

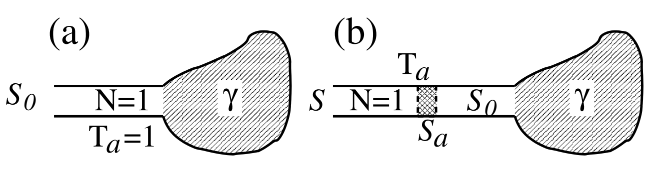

FIG. 1. Sketch of the model for scattering with direct processes and absorption. In (a) S_0 describes the scattering of a billiard with absorption γ but perfect coupling to the lead ($T_a = 1$). In (b) we associate the S matrix S_a given by Eq. (5) to the barrier that describes the nonideal coupling. The resulting S matrix of the system with absorption γ and coupling T_a can be written in terms of S_0 and S_a.

$$S_a = \begin{pmatrix} -\sqrt{1-T_a} & \sqrt{T_a} \\ \sqrt{T_a} & \sqrt{1-T_a} \end{pmatrix}. \tag{5}$$

The combination rule of S matrices gives the following relation between S and S_0:

$$S_0(S) = \frac{S - \langle S \rangle}{1 - \langle S \rangle S}, \tag{6}$$

where $\langle S \rangle = \sqrt{1 - T_a}$ [see Eq. (3)]. Then the distribution $P_S(S)$ for the system including direct processes is $P_{S,0}(S_0)$ times the Jacobian of the transformation (6). The result is

$$\begin{aligned} P_S(S) &= \left| \frac{\partial(R_0, \theta_0)}{\partial(R, \theta)} \right| P_{S,0}(S_0), \\ &= \left(\frac{1 - \langle S \rangle^2}{|1 - S\langle S \rangle|^2} \right)^2 \frac{1}{2\pi} P_{R,0}(R_0), \end{aligned} \tag{7}$$

where $R_0 = |S_0|^2$. The distribution $P_{R,0}(R_0)$ is known for several cases [11,12]:

$$P_{R,0}(R_0) = \begin{cases} \frac{\exp[-\alpha/(1-R_0)]}{(1-R_0)^3}[A + B(1 - R_0)], & \beta = 2, \\ Ce^{-\alpha/(1-R_0)}/(1 - R_0)^{2+\beta/2}, & \gamma \ll 1, \\ \alpha e^{-\alpha R_0}, & \gamma \gg 1, \end{cases} \tag{8}$$

where $\alpha = \gamma\beta/2$, $A = \alpha(e^\alpha - 1)$, $B = (1 + \alpha - e^\alpha)$, and $C = \alpha^{1+\beta/2}/\Gamma(1 + \beta/2)$. In the first case of Eq. (8)

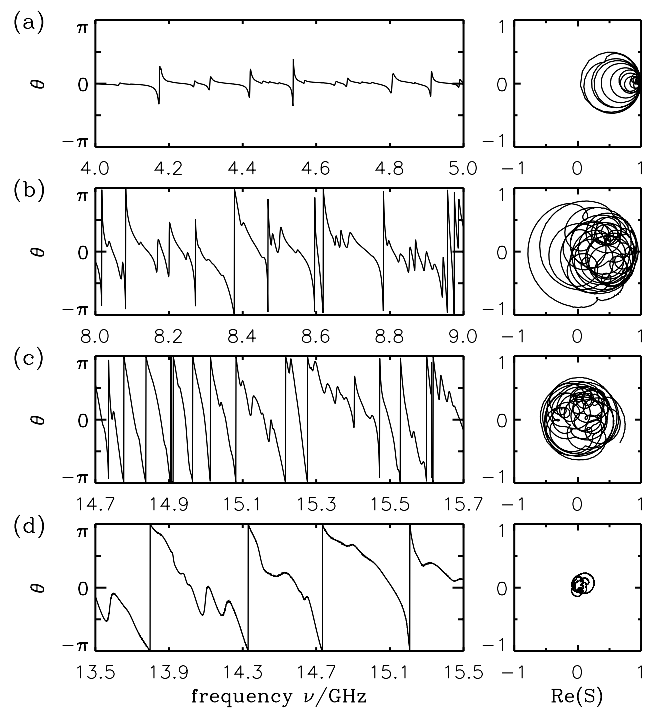

FIG. 2. The phase θ of the S matrix for different coupling and absorption regimes: (a) $\gamma = 0.56$, $T_a = 0.116$ (weak absorption, weak coupling), (b) $\gamma = 2.42$, $T_a = 0.754$, (c) $\gamma = 8.40$, $T_a = 0.989$, and (d) $\gamma = 48.00$, $T_a = 0.998$ (strong absorption, nearly perfect coupling). On the right-hand side, the Argand diagrams show Im(S) versus Re(S).

the absorption can take any value, whereas in the other cases β can take any value. As our experimental results are for a time-reversal system, we need the distribution for $\beta = 1$. An approximate distribution for $\beta = 1$ was given in Ref. [13], which is valid, however, for intermediate and strong absorption only. The interpolation formula

$$P_{R,0}(R_0) = C_\beta \frac{e^{-\alpha/(1-R)}}{(1-R)^{2+\beta/2}}[A\alpha^{\beta/2-1} + B(1-R)^{\beta/2}], \quad (9)$$

with $C_\beta = [A\Gamma(1+\beta/2,\alpha)/\alpha^2 + Be^{-\alpha}/\alpha]^{-1}$, satisfies all cases of Eq. (8). Here $\Gamma(x,\alpha) = \int_\alpha^\infty t^{x-1}e^{-t}dt$ is the upper incomplete Gamma function. The deviation between the interpolation formula Eq. (9) and numerical simulations ($\beta = 1$) are of the order of a few percent. A number of tests had been performed in [24]. In all cases a good agreement was found. Notice that the theory presented here is rigorous for the imperfect coupling or direct processes; only the ansatz given by Eq. (9) for the absorption is heuristic.

The experimental setup was described in Ref. [19]. The following two systems were investigated: (i) a half Sinai billiard and (ii) a half Sinai billiard with a microwave absorber attached to one side. To improve statistics, in both cases the semicircle was moved along the wall in steps of 5 mm to obtain more than 50 measurements. The complex S matrix was measured with a vector network analyzer. We investigated four different regimes also considered in Ref. [19] ranging from weak absorption and weak coupling ($\gamma = 0.56$, $T_a = 0.116$) to strong absorption and nearly perfect coupling ($\gamma = 48$, $T_a = 0.998$). The one-channel case is realized due to the fact that only a single antenna is attached with a radius much smaller than the wavelength.

In Fig. 2 we plot the phase θ of S as a function of the frequency and the corresponding Argand diagrams of the scattering matrix S for different coupling and absorption regimes. The experimental value of T_a is obtained directly from the mean value of the S matrix by Eq. (3). The measured value of $\langle R\rangle$ then fixes the absorption strength γ, which decreases monotonically as a function of $\langle R\rangle$ and vice versa.

To test the theoretical and experimental results, we made numerical simulations based on random-matrix theory. We expressed the S matrix as

$$S(E) = 1 - 2\pi i W^\dagger (E - H + i\pi W W^\dagger)^{-1} W, \quad (10)$$

where H is taken from the Gaussian orthogonal ensemble and W gives the coupling between the resonant modes of the cavity and the channel states. E is related to the electromagnetic frequency ν by Weyl's formula. More details can be found in Refs. [19,25].

In Fig. 3 we plot the experimental distributions of the S matrix for the same cases as in Fig. 2. The numerical distributions are also included in order to test the theoretical ones based on Eqs. (4), (7), and (9). An excellent agreement between theory, numerics, and experiment is found. Finally, in Fig. 4 we show the distribution $P_\theta(\theta)$ for the different regimes of absorption and antenna coupling. The numerical simulation is included as well as the theoretical results obtained by integrating Eq. (7) numerically.

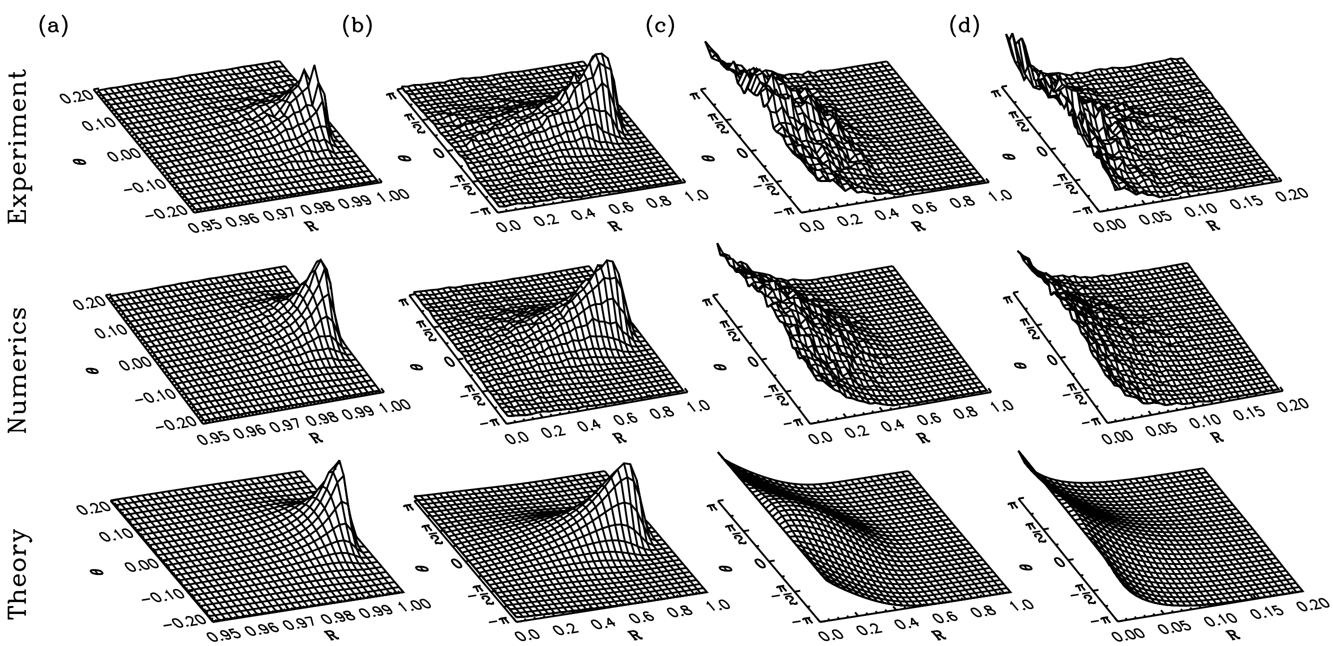

FIG. 3. Distribution of the S matrix for the same ranges of coupling and absorption of Fig. 2. The upper, central, and lower rows correspond to experiment, numerics, and theory, respectively. Note the change of scales for R and θ.

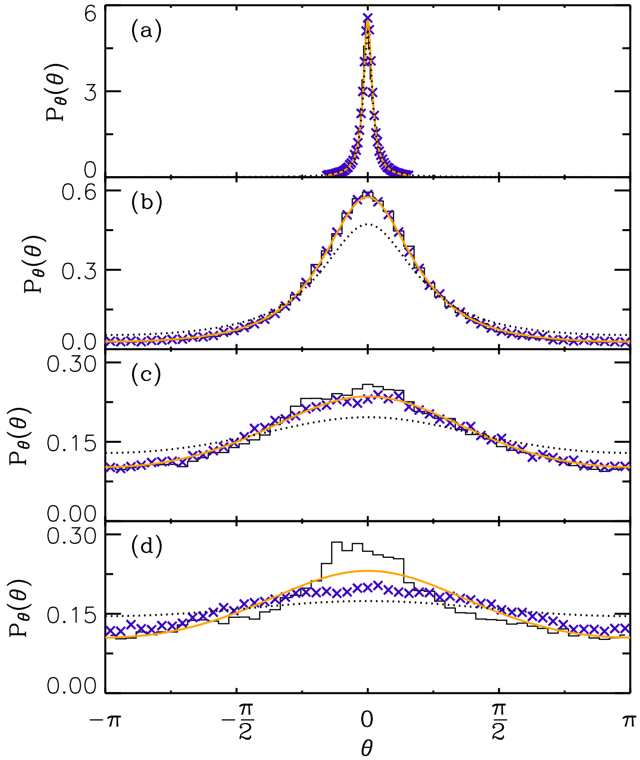

FIG. 4 (color online). Experimental distribution $P_{\theta,T}(\theta)$ (histogram) for the same regimes of coupling and absorption as given in Fig. 2. Additionally, the numerical (crosses) and the theoretical (solid line) is plotted as well. The dotted curves correspond to the Poisson kernel without absorption [see Eq. (1)].

Again an excellent agreement is found for Figs. 4(a)–4(c). The deviations in the case of strong absorption and nearly perfect coupling $T_a = 0.998$ [Fig. 4(d)] are due to the fact that the distribution $P_\theta(\theta)$ depends very sensitively on the coupling parameter T_a. Already a slight change of T_a to 0.996 improves the agreement between theory and experiment considerably. In addition, the coupling T_a changes within the investigated frequency range, which was not taken into account in the theory.

In conclusion, we have shown experimental evidence of direct processes in the scattering of chaotic microwave cavities in the presence of absorption. This was done by (i) obtaining the experimental distribution of the S matrix for various regimes of absorption and antenna coupling in the one-channel case, (ii) obtaining the same distributions numerically, and (iii) presenting a theory that generalizes Poisson's kernel to include absorption. Excellent agreement between theory, numerics, and experiment was obtained in all regimes of absorption and antenna coupling. Very recently an exact expression for $P_{R,0}(R_0)$ has been derived deviating only slightly from the approximation given in Eq. (9) [26]. Direct processes can thus be determined by deviations from the uniform distribution of the phases even in the presence of absorption.

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