Synchronization in Complex Networks with Age Ordering

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The propensity for synchronization is studied in a complex network of asymmetrically coupled units, where the asymmetry in a given link is determined by the relative age of the involved nodes. In growing scale-free networks, synchronization is enhanced when couplings from older to younger nodes are dominant. We describe the requirements for such an effect in a more general context and compare with the situations in nongrowing random networks with and without a degree ordering.

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From the brain over the Internet to human society, complex networks are prominent candidates to describe sophisticated collaborative dynamics in many areas [1]. Of particular interest are the so called small-world (SW) and scale-free (SF) wirings. SWs are intermediate wirings between regular lattices (RLs) and random networks (RNs) [2]. They are characterized by a path length ℓ scaling logarithmically with the network size N ($\ell \propto \log N$, in contrast to the linear scaling of RLs), yet with a clustering structure much higher than a RN. A specific example of a SW is the SF configuration, in which the degree k (the number of edges in a node) follows a power law distribution $p(k) \sim k^{-\gamma}$. A SF network can be grown by adding successively new nodes to the network and connecting them with the already existing ones by the preferential attachment rule [3].

Recently, the dynamics of complex networks has been extensively investigated with regard to collective (synchronized) behaviors [4], with special emphasis on the interplay between the complexity in the overall topology and the local dynamical properties of the coupled units. It has been observed that the SW property can increase the propensity for synchronization (PFS) in networks of Hodgkin-Huxley neurons or phase oscillators [5], while, on the other hand, synchronization can become more difficult as the heterogeneity of a SF network increases [6]. The theoretical framework for determining the PFS has been established by Barahona and Pecora [7], who separated the topological part of the problem from the part involving the dynamics on the local nodes.

A basic assumption of previous works is that the local units are symmetrically coupled with uniform undirected coupling strengths (unweighted links). This simplification, however, does not match satisfactorily the peculiarities of many real networks. In ecological systems, for instance, the nonuniform weight in prey-predator interactions plays a crucial role in determining the food web dynamics [8]. Similarly, the interaction between individuals in social networks [9] is never symmetric, rather it depends upon several social factors, such as age, social class or influence, personal leadership, or charisma.

In this Letter, we analyze networks of asymmetrically coupled dynamical units. By explicitly relating the asymmetry in the connections to an age order among different nodes, we will give evidence that age ordered networks provide a better PFS. In particular, we will show that the three main ingredients maximizing PFS are (i) heterogeneity in the network topology allowing for the existence of nodes with very large degrees (hubs) together with nodes with very small degrees (nonhubs), (ii) asymmetry in the connections forcing a preferential coupling direction from hubs to nonhubs, and (iii) a structure of connected hubs in the network.

We here adopt the idea that the direction of an edge can be determined by an age ordering between the connected nodes. For growing networks (such as SF) the age order is naturally related to the appearance order of the node during the growing process. We consider a network of N linearly coupled identical systems. The equation of motion reads

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij}\mathbf{H}[\mathbf{x}_j], \qquad i = 1, \dots N, \tag{1}$$

where $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ governs the local dynamics of the vector field $\mathbf{x}_i$ in each node, $\mathbf{H}[\mathbf{x}]$ is a linear vectorial function, and σ is the coupling strength. G_{ij} is a zero row-sum coupling matrix with off diagonal entries $G_{ij} = -\mathcal{A}_{ij}\frac{\Theta_{ij}}{\sum_{j\in K_i}\Theta_{ij}}$, where $\mathcal{A}$ is the adjacency matrix, and $\Theta_{ij} = \frac{1-\theta}{2}$ ($\Theta_{ij} = \frac{1+\theta}{2}$) for $i > j$ ($i < j$). K_i is the set of k_i neighbors of the ith node, and the parameter $-1 < \theta < 1$ governs the coupling asymmetry in the network. Precisely, the limit $\theta \to -1$ ($\theta \to +1$) gives a unidirectional coupling where the old (young) nodes drive the young (old) ones. Asymmetric coupling was also recently established for nonidentical space extended fields [10].

Following the ideas of Ref. [7], the network PFS can be inspected by linear stability of the synchronous state

($\mathbf{x}_i = \mathbf{x}_s$, $\forall i$). By diagonalizing the variational equation, one obtains N blocks of the form $\dot{\zeta}_i = J\mathbf{F}(\mathbf{x}_s)\zeta_i + \sigma\lambda_i\mathbf{H}[\zeta_i]$, which differ only by the eigenvalues of the coupling matrix λ_i (here J is the Jacobian operator). The behavior of the largest Lyapunov exponent associated with $\nu = \sigma\lambda_i$ (also called master stability function [11]) fully accounts for the linear stability of the synchronization manifold. Namely, the synchronous state (associated with $\lambda_1 = 0$) is stable if all the remaining blocks, associated with λ_i ($i \geq 2$), have negative Lyapunov exponents.

For a generic θ, our coupling matrix is asymmetric, and therefore its spectrum is contained in the complex plane ($\lambda_1 = 0$; $\lambda_l = \lambda_l^r + j\lambda_l^i$, $l = 2, \ldots, N$). Moreover, since all elements of G are real, nonreal eigenvalues appear in pairs of complex conjugates. In the following, we will order the eigenvalues of G for increasing real parts. Gerschgorin's circle theorem [12] asserts that G's spectrum is fully contained within the union of circles (C_i) having as centers the diagonal elements of G (d_i), and as radii the sums of the absolute values of the other elements in the corresponding rows ($\{\lambda_l\} \subset \cup_i C_i[d_i, \sum_{j\neq i} \mid G_{ij} \mid]$).

By construction, the diagonal elements of G are normalized to 1 in all possible cases. It is crucial to emphasize the physical and mathematical relevance of this choice. Physically, this normalization prevents the coupling term from being arbitrarily large (or arbitrarily small) for all possible network topologies and sizes, thus making it a meaningful realization of what happens in many real world situations (such as neuronal networks) where the local influence of the environment on the dynamics does not scale with the number of connections. Mathematically, since G is a zero row-sum matrix (and furthermore $d_i = \sum_{j\neq i} \mid G_{ij} \mid$ because all nonzero off diagonal elements are negative), this warrants *in all cases* that G's spectrum is fully contained within the unit circle centered at 1 on the real axis ($\mid \lambda_l - 1 \mid \leq 1$, $\forall l$), giving the following inequalities: (i) $0 < \lambda_2^r \leq \ldots \leq \lambda_N^r \leq 2$, and (ii) $\mid \lambda_l^i \mid \leq 1$, $\forall l$. This latter property is essential to provide a consistent and unique mathematical framework within which one can formally assess the relative merit of one topology against another for optimal PFS.

Let $\mathcal{R}$ be the bounded region in the complex plane where the master stability function provides a negative Lyapunov exponent. The stability condition for the synchronous state is that the set $\{\sigma\lambda_l, l = 2, \ldots, N\}$ be entirely contained in $\mathcal{R}$ for a given σ. The best PFS is then assured when both the ratio $\frac{\lambda_N^r}{\lambda_2^r}$ and $M \equiv \max_l\{\mid \lambda_l^i \mid\}$ are simultaneously made as small as possible.

We start with analyzing the effects of heterogeneity in the node degree distribution, by comparing the PFS of a class of SF networks with different degree distributions with that of a highly homogeneous RN. The used class of SF networks is obtained by a generalization of the preferential attachment growing procedure [3]. Namely, starting from $m + 1$ all to all connected nodes, at each step a new node is added with m links, connecting to old nodes with probability $p_i = \frac{k_i + B}{\sum_j (k_j + B)}$ (k_i being the degree of the ith node, and B a tunable real parameter, representing the initial attractiveness of each node). The γ exponent of the power law scaling in the degree distribution $p(k) \sim k^{-\gamma(B,m)}$ is then given by $\gamma(B, m) = 3 + \frac{B}{m}$ in the thermodynamic ($N \to \infty$) limit [13]. While the average degree is by construction $\langle k \rangle = 2m$ (thus independent of B), the heterogeneity of the degree distribution is strongly modified by B. This induces convergence of higher order moments of $p(k)$, at variance with the case $B = 0$ recovering the model of Ref. [3].

For comparison, a highly homogeneous Erdös-Rényi RN [14] is considered, with connection probability $P = \frac{2m}{N-1}$ (giving same average degree $\langle k \rangle = 2m$), with an arbitrary initial age ordering. Figure 1(a) shows λ_N^r and λ_2^r vs θ for a SF network with $m = 5$ and $B = 0$ (solid line) and for the chosen RN (dashed line). All calculations refer to ensemble averages over 24 different realizations of networks with 500 nodes. For RNs, the curve $\lambda_N^r(\theta)$ [$\lambda_2^r(\theta)$] displays a minimum [maximum] for $\theta = 0$, showing that asymmetry here deteriorates the network PFS. At variance, for a SF network the difference between λ_N^r and λ_2^r continuously shrinks, as θ decreases. As a consequence, Fig. 1(b) reports the behavior of the eigenratio $\frac{\lambda_N^r}{\lambda_2^r}$, making it clear that while the best PFS in RN is obtained for $\theta = 0$, SF shows better (worse) PFS for $\theta \to -1$ ($\theta \to 1$). As for the imaginary part of the spectra, Fig. 1(c) reports M vs θ, indicating only very small differences between SF and RN in the whole range of the asymmetry parameter, and highlighting that the contribution to the PFS of the imaginary part of the spectra does not depend significantly on the specific network structure. These findings have been consistently observed in subsequent results.

The second step of our study is the investigation of the PFS in SF at different m and B values. In Fig. 2(a), PFS is compared for different m for $B = 0$. Since m determines

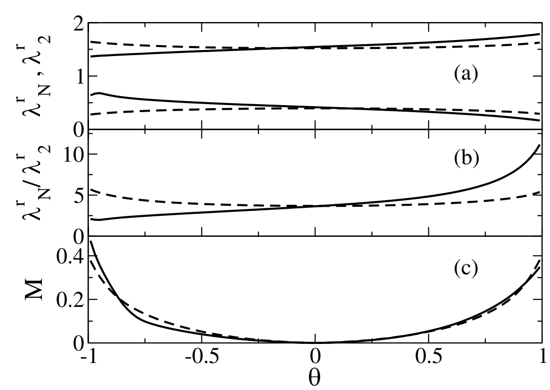

FIG. 1. λ_N^r and λ_2^r (a), $\frac{\lambda_N^r}{\lambda_2^r}$ (b), and M (c) vs θ for a SF network ($m = 5$ and $B = 0$, solid lines) and RNs (dashed lines). All quantities and all other network parameters are specified in the text.

the average degree, as m increases, the average connectivity increases and synchronizability is naturally enhanced. The most important point is that the monotonically decreasing behavior of $\frac{\lambda_N^r}{\lambda_2^r}$ with θ persists for all m, indicating that synchronization is always enhanced when θ becomes smaller. In Fig. 2(b), PFS is compared for $m = 5$ and for various values of B, altering the exponent of the degree distribution. For all B values, almost the same enhancement for negative θ is observed. In Ref. [6] it was found that, for symmetric coupling, the PFS does decrease as the network becomes more heterogeneous, due to an overload of the traffic of communication passing through the highly connected nodes. However, since in our model the input of each node is normalized to 1, this overload effect does not take place, and there is even a slightly enhanced PFS with decreasing B, as can be seen from the inset of Fig. 2(b). Therefore one can conclude that in aged growing networks PFS depends on the average degree, but the asymmetry enhanced synchronization phenomenon does not significantly depend on heterogeneity.

This leads us to discuss the main point of our study, concerning the determination of the essential topological ingredients enhancing PFS in weighted (aged) networks. The first ingredient is that the weighting must induce a dominant interaction from hub to nonhub nodes. This can be easily understood by a simple example: the case of a star network consisting of a single large hub (the center of the star) and several nonhub nodes connected to the hub. When the coupling is dominant from nonhub to hub nodes, synchronization can be prevented by the fact that the hub receives a set of independent inputs from the different nonhub nodes. In the reverse case (when the center drives the periphery of the star) synchronization can be easily achieved. The very same mechanism occurs in our SF case. Indeed, for positive (negative) θ values, the dominant coupling direction is from younger (older) to older (younger) nodes. Now, in SF the minimal degree of a node is by construction m and older nodes are more likely to display larger degrees than younger ones, so that a negative θ here induces a dominant coupling direction from hubs to nonhub nodes.

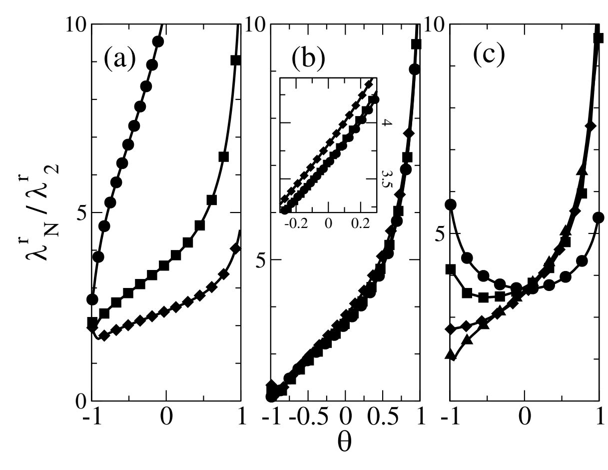

FIG. 2. $\frac{\lambda_N^r}{\lambda_2^r}$ vs θ for (a) SF with $B = 0$ and $m = 2$ (circles), 5 (squares), and 10 (diamonds); (b) SF with $m = 5$ and $B = 0$ (circles), 5 (squares), and 10 (diamonds); (c) RNs with arbitrary age order (circles), RNs with age depending on degree (squares), SF with $m = 5$ and $B = 0$ without connection between nodes 1 and 5 (diamonds), and SF with $m = 5$ and $B = 0$ (triangles). The inset of (b) is a zoom of the region close to $\theta = 0$.

The second ingredient is that the network contains a structure of connected hubs influencing the other nodes. In our SF case, the normalization in the off diagonal elements of G [15] assures that hubs receive an input from a connected node scaling with the inverse of their degree.

In order to make evident the validity of these claims, we have considered the PFS of a series of *ad hoc* modified networks. The results are summarized in Fig. 2(c). First, we have reordered the node age in RNs according to each node degree. The resulting $\frac{\lambda_N^r}{\lambda_2^r}(\theta)$ (curve with squares) shows now a minimum for $\theta \approx -0.5$, in contrast to the case with arbitrary aging (curve with circles). This confirms the need of a dominant interaction from hubs to nonhubs for improving PFS, also for highly homogeneous networks. As for the second ingredient, starting from a SF with $m = 5$ and $B = 0$ (curve with triangles), we artificially disconnected the initially existing link between the first and fifth network nodes. These are indeed the two hubs with the highest degree in the SF configuration. The result is shown in the curve with diamonds, where one sees that such a small perturbation (the difference in the two networks is limited to only a link) is already sufficient to substantially weaken the PFS. The situation remains better, however, than the two RN cases, indicating that the structure of growing aged network inherently enhances synchronization.

Finally, we illustrate our arguments by an example of a network of coupled chaotic Rössler oscillators [16]. The dynamics is ruled by Eq. (1), with $\mathbf{x} = (x, y, z)$, $\mathbf{F}(\mathbf{x}) = [-y - z, x + 0.165y, 0.2 + z(x - 10)]$, and $\mathbf{H}[\mathbf{x}] = x$. The master stability function is depicted in Fig. 3(d) in the complex plane $\nu = \alpha + j\beta$. The bold solid line (denoting a zero Lyapunov exponent) is the boundary between stability ($\mathcal{R}$) and instability regions for the synchronization manifold. Figures 3(a)–3(c) report the location of the G spectrum for a SF network for $m = 2$, $B = 0$, and $N = 500$ and with $\theta = -0.8$, 0, and 0.8, respectively. While for $\theta = 0$ the spectrum is real, comparison of panels (a) and (c) shows that the spectra at negative θ values are much less dispersed in the complex plane, thus increasing the range of σ values for which synchronization can be achieved in the network.

The appearance of the synchronous state can be monitored by looking at the vanishing of the time average (over a window T) synchronization error $\langle D \rangle = \frac{1}{T(N-1)} \sum_{j>1} \int_t^{t+T} \|\mathbf{x}_j - \mathbf{x}_1\| dt'$. In the present case, we

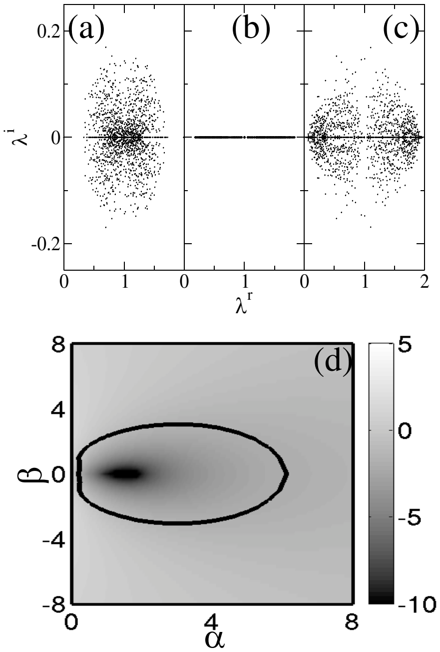

FIG. 3. Distribution of G's eigenvalues in the complex plane for a SF network with $m = 2$, $B = 0$, and $\theta = -0.8$ (a), $\theta = 0$ (b), and $\theta = 0.8$ (c). (d) Master stability function in the complex plane for coupled Rössler oscillators.

adopt as vector norm $\|\mathbf{x}\| = |x| + |y| + |z|$. Figure 4 reports $\langle D \rangle$ vs θ for RNs with arbitrary age (a) and for a SF network with $m = 2$ and $B = 0$ (b). The curves with circles, squares, diamonds, up-pointing triangles, and left-pointing triangles refer to $\theta = -0.8$, -0.4, 0, 0.4, and 0.8, respectively. While in the RN case the range for synchronization is substantially independent on θ [reflecting the behavior of $\frac{\lambda_N^r}{\lambda_2^r}$ in the dashed line of Fig. 1(b)], the case with SF [to be compared with the curve with circles in Fig. 2(a)] confirms that synchronization is strongly affected by the asymmetry. In particular, negative (positive) values of θ have the effect of increasing (decreasing) the range of coupling strengths over which synchronization occurs with respect to the case $\theta = 0$.

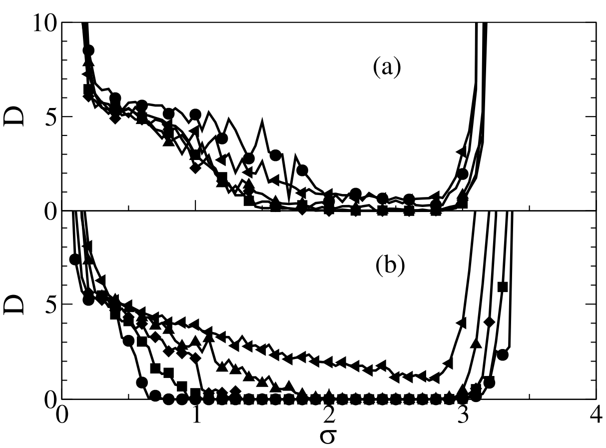

FIG. 4. $\langle D \rangle$ (see text for definition) vs σ for RNs (a) and for a SF network [$m = 2$ and $B = 0$ (b)]. In both graphs the curves with circles, squares, diamonds, up-pointing triangles, and left-pointing triangles refer to $\theta = -0.8$, -0.4, 0, 0.4, and 0.8, respectively.

In conclusion, we have demonstrated that the PFS is enhanced in networks of asymmetrically coupled units. In growing SF such enhancement is particularly evident when the dominant coupling direction is from older to younger nodes. A key aspect of social organizations is the dynamics of information exchange. Our approach may provide new insights in the study of collective communication or coordination in distributed social networks, as well as useful hints for understanding the formation of social collective behaviors (leading opinions, rumors, political orientations, dominant tastes, habits, fashion, etc.).

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