Derivation of the Order Parameter of the Chiral Potts Model

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We derive the order parameter of the chiral Potts model, using the method of Jimbo *et al*. The result agrees with previous conjectures.

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The solvable chiral Potts model is one of N-state spins with nearest-neighbor interactions on a planar lattice $\mathcal{L}$ [1]. It is like other solvable models in that its interactions satisfy the star-triangle relations. However, it is unlike most of them in that it does not have the "rapidity difference" property. This makes the model mathematically much more difficult. The free energy of the model has been obtained using functional transfer matrix relations [2–4], but the order parameters (spontaneous magnetizations) have so far defied calculation. In particular, the corner transfer matrix method appears to fail completely [5].

This is despite the fact that there is an elegant and eminently believable conjecture for these order parameters [6–8].

Here we derive the order parameters and verify the conjecture. The method is quite similar to the "inversion relation" method for calculating the free energy [9]. Both involve assuming certain analyticity properties of a "$\tau_2(t_q)$" model (which is closely related to the superintegrable chiral Potts model).

We take $\mathcal{L}$ to be the square lattice, drawn diagonally as in Fig. 1. Each spin i takes one of N possible states, labeled $0, \ldots, N-1$. It interacts with a neighboring site j with Boltzmann weight $W_{vh}(\sigma_i - \sigma_j)$ or $\overline{W}_{vh}(\sigma_i - \sigma_j)$ if i is below j and the edge is in the SW-NE or SE-NW direction, respectively.

Here v, h each denote a set of rapidity variables. Let $p = \{x_p, y_p, \mu_p, t_p\}$, where x_p, y_p, μ_p, t_p are related to one another by $t_p = x_p y_p$,

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N), \qquad k x_p^N = 1 - k'/\mu_p^N,$$
$$k y_p^N = 1 - k'\mu_p^N, \qquad k^2 t_p^N = (1 - k'\mu_p^N)(1 - k'/\mu_p^N) \tag{1}$$

and

$$k = (1 - k'^2)^{1/2}, \qquad 0 < k, k' < 1. \tag{2}$$

Here k, k' are constants, the same at all sites of the lattice, and k' is a "temperaturelike" variable, being small at low temperatures. Here we take $0 < k' < 1$, which means that the system is ferromagnetically ordered. It becomes critical when $k' \to 1$.

Rapidities such as p, q, v, h are associated with the dotted lines of Fig. 1 and may vary from line to line. If the edges (i, j), (i, k) of $\mathcal{L}$ are intersected by rapidity lines v, h, ordered as in Fig. 1, then the edge weight functions are $W_{vh}(\sigma_i - \sigma_j)$, $\overline{W}_{vh}(\sigma_i - \sigma_k)$ where

$$W_{pq}(n) = (\mu_p/\mu_q)^n \prod_{j=1}^{n} \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}, \tag{3}$$

$$\overline{W}_{pq}(n) = (\mu_p \mu_q)^n \prod_{j=1}^{n} \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p}, \tag{4}$$

where $\omega = \exp(2\pi i/N)$ and $v = \{x_v, y_v, \mu_v, t_v\}$, $h = \{x_h, y_h, \mu_h, t_h\}$ are rapidity sets satisfying (1).

If $x_p, x_q, y_p, y_q, \omega x_p$ all lie on the unit circle in an anticlockwise sequence, then $W_{pq}(n)$, $\overline{W}_{pq}(n)$ are real and positive. We refer to this as the physical case.

Hold the spin a in Fig. 1 fixed. Then the partition function is

$$Z(a) = \sum_{\sigma} \prod W_{vh}(i - j) \prod \overline{W}_{vh}(j - k), \tag{5}$$

the products being over all edges of each type (with their appropriate rapidity variables), and the sum over all values of the other spins, the boundary spins being set to zero. The unrestricted partition function is $Z = Z(0) + \cdots + Z(N-1)$.

We have followed Jimbo *et al.* [10] and cut the horizontal rapidity line immediately below a, giving the left (right) half line a rapidity p (q). Define

$$F_{pq}(a) = Z(a)/Z. \tag{6}$$

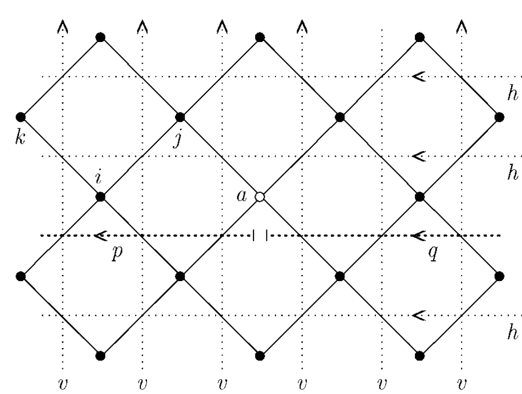

FIG. 1. The square lattice with the cut rapidity line below a.

We expect this ratio to tend to a limit when the lattice is large. It will *not* depend on the "background" rapidities v, h because of the star-triangle relation [1], which allows us to move any of the v, h rapidity lines infinitely far away from the spin a.

However, we cannot move the half lines p, q away from a, so $F_{pq}(a)$ will indeed be a function of p, q. It is the probability that the central spin has value a. An exceptional case is when $q = p$, when the cut disappears and the recombined line can be moved to infinity, so

$$F_{pp}(a) = \text{ independent of } p. \tag{7}$$

What we can do is rotate p, q round a and then interchange them, which gives us some functional relations satisfied by $F_{pq}(a)$ [11]. If we define

$$\tilde{F}_{pq}(r) = \sum_{a=0}^{N-1} \omega^{ra} F_{pq}(a), \tag{8}$$

$$G_{pq}(r) = \tilde{F}_{pq}(r)/\tilde{F}_{pq}(r-1), \tag{9}$$

then for the infinite lattice we obtain the relations

$$G_{Rp,Rq}(r) = 1/G_{pq}(N-r+1), \tag{10a}$$

$$G_{pq}(r) = G_{Rq,R^{-1}p}(r), \tag{10b}$$

$$G_{pq}(r) = \frac{x_q\mu_q - x_p\mu_p\omega^r}{y_p\mu_q - y_q\mu_p\omega^{r-1}} G_{R^{-1}q,Rp}(r), \tag{10c}$$

$$\prod_{r=1}^{N} G_{pq}(r) = 1, \tag{10d}$$

$$G_{Mp,q}(r) = G_{p,M^{-1}q}(r) = G_{pq}(r+1). \tag{10e}$$

Here R, M are automorphisms that act on the rapidities:

$$\{x_{Rp}, y_{Rp}, \mu_{Rp}, t_{Rp}\} = \{y_p, \omega x_p, 1/\mu_p, \omega t_p\}. \tag{11}$$

$$\{x_{Mp}, y_{Mp}, \mu_{Mp}, t_{Mp}\} = \{x_p, y_p, \omega\mu_p, t_p\}. \tag{12}$$

We regard t_p as an independent complex variable and x_p, y_p, μ_p as determined from it by (1). They are multivalued functions of t_p: to make them single-valued we must cut the t_p plane as in Fig. 2. There are N cuts $C_0, \ldots, C_{N-1}$, where C_j lies on the radial line $\arg(t_p) = 2\pi j/N$.

The case we shall be interested in is when $|\mu_p| > 1$ and $\arg(x_p)$ is between $-\pi/(2N)$ and $\pi/(2N)$. Then x_p lies in a small approximately circular region $\mathcal{R}_0$ round $x_p = 1$, while y_p lies in a plane with N corresponding approximately circular holes $\mathcal{R}_0, \ldots, \mathcal{R}_{N-1}$ surrounding the points $1, \omega, \ldots, \omega^{N-1}$. We say that p lies in the domain $\mathcal{D}_1$.

In the low temperatures limit $k' \to 0$ and C_j, $\mathcal{R}_j$ both shrink to the point ω^j. Then $x_p \to 1$ and y_p can lie anywhere except at a root of unity. We take $p, q, h \in \mathcal{D}_1$ and

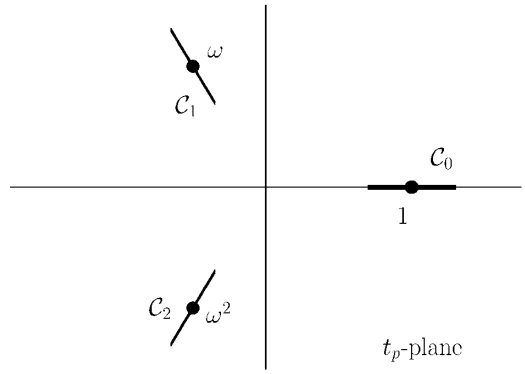

FIG. 2. The t_p plane for $N = 3$, showing the cuts C_0, C_1, C_2.

$\mu_v = \mathrm{O}(1)$, $x_v \simeq y_v \simeq 1$. Then $W_{vh}(n)$, $\overline{W}_{vh}(n)$ are small unless $n = 0 \pmod N$, which is the usual low-temperature case. The sum in (5) is dominated by the contribution from all spins other than a being zero, and we obtain

$$F_{pq}(a) = \frac{k'^2(\mu_p/\mu_q)^a}{N^2(1-\omega^a)(1-\omega^{-a})} \prod_{j=1}^{a} \frac{1-\omega^{j-1}t_q}{1-\omega^{j-1}t_p}. \tag{13}$$

Let us take q to be related to p by

$$x_q = x_p, \qquad y_q = \omega y_p, \qquad \mu_q = \mu_p. \tag{14}$$

Then (13) simplifies to

$$F_{pq}(a) = \frac{k'^2}{N^2(1-\omega^a)(1-\omega^{-a})} \frac{1-\omega^a t_p}{1-t_p} + \mathrm{O}(k'^4). \tag{15}$$

Thus to leading order in k' we see that $F_{pq}(a)$ is an analytic and bounded function of t_p except at $t_p = 1$. This is consistent with $F_{pq}(a)$ and $G_{pq}(r)$ having branch cuts (for nonzero k') on C_0, but there is no indication of branch cuts at $C_1, \ldots, C_{N-1}$.

We argue that this is exactly true for sufficiently small nonzero values of k'. If we rotate and reverse the left half line p anticlockwise below a, as in [11], it becomes a left-pointing line with rapidity $p' = R^{-1}p$ [12]. (To ensure that at low temperatures the dominant contribution continues to come from all noncentral spins being zero, we should reduce μ_v to order k', μ_h to order unity, and take $x_h \simeq y_h \simeq 1$.)

The half line p' now lies immediately below q, and from (14)

$$x_q = y_{p'}, \qquad y_q = \omega^2 x_{p'}, \qquad \mu_q = 1/\mu_{p'}. \tag{16}$$

This is precisely the condition for the combined weights of the two half rows p', q (summed over intervening spins) to be those of the $\tau_2(t_{p'})$ model [Eqs. (2.38)–(3.48) of [2], with $k = 0$, $\ell = 2$]. For a finite lattice, if there are s spins to the left of a, then $Z(a)$ depends on p only via t_p, and is a *polynomial* of degree s in t_p.

Further, when k' is small and $a = 0$, this polynomial is $(1 - t_p)^s$, and is small (of order k'^2) when $0 < a < N$. Hence $G_{pq}(r)$ is the ratio of two polynomials, each of degree s and tending to $(1 - t_p)^s$ as $k' \to 0$.

By continuity, for sufficiently small but nonzero k' the zeros of the polynomials must be close to $t_p = 1$, i.e., $t_{p'} = 1/\omega$.

This is the same behavior as the free energy of the $\tau_2(t_{p'})$ model. This leads us *assume*, corresponding to Assumption 2 of [9], that $G_{pq}(r)$ is analytic in the cut t_p plane of Fig. 2, except only for the branch cut C_0 [13].

Now look at (10c). Using (14) it becomes, for $r = 1, \ldots, N - 1$,

$$x_p^{-1} G_{pq}(r) = x_{\overline{p}}^{-1} G_{\overline{p},\overline{q}}(r), \tag{17}$$

where $\overline{p} = R^{-1}q, \overline{q} = Rp$. It follows that $\overline{p}, \overline{q}$ satisfy the relation (14) and $x_{\overline{p}} = y_p, y_{\overline{p}} = x_p, \mu_{\overline{p}} = 1/\mu_p$. The left-hand side of this equation is therefore the same as the right-hand side, with the same value of t_p, but with x_p, y_p interchanged and μ_p inverted.

This is what happens if one crosses the branch cut C_0 in Fig. 2 and then returns to the original value of t_p. Thus (17) is equivalent to the statement that $x_p^{-1} G_{pq}(r)$ does *not* have a branch cut at C_0.

Write $G_{pq}(r)$ as $G_r(t_p)$ and consider the function

$$L_r(t_p) = x_p^{-1} G_r(t_p) G_r(\omega t_p) \cdots G_r(\omega^{N-1} t_p). \tag{18}$$

For $r \neq 0$ the factor $x_p^{-1} G_r(t_p)$ has no cut on C_0. From our assumption, neither do any of the other G factors. Hence $L_r(t_p)$ has no such cut. It is unchanged by $t_p \to \omega t_p$, so it has no cuts at any of the C_j. When $t_p, y_p, \mu_p \to \infty$ (their ratios remaining finite and nonzero), the Boltzmann weights remain finite, so we expect $L_r(t_p)$ to be bounded at infinity. From Liouville's theorem it is therefore a constant, so

$$L_r(t_p) = C_r, \qquad 0 < r < N, \tag{19}$$

C_r being a constant [14].

When $t_p, y_p = 0$, then $q = p$ and $x_p^N = k$. If $t_p, y_p, \mu_p \to \infty$, then $x_p^N = 1/k$ and $q = M^{-1}p$. From (7) and (10e) it follows that

$$C_r = k^{-1/N} G_{pp}(r)^N = k^{1/N} G_{pp}(r + 1)^N, \tag{20}$$

for $r = 1, \ldots, N - 1$.

Eliminating C_r and using (10d), we obtain

$$G_{pp}(r) = k^{(N+1-2r)/N^2} \tag{21}$$

for $r = 1, \ldots, N$.

The order parameter of the chiral Potts model is

$$\langle \omega^{ra} \rangle = \tilde{F}_{pp}(r)/\tilde{F}_{pp}(0) = G_{pp}(1) \cdots G_{pp}(r), \tag{22}$$

so we have

$$\langle \omega^{ra} \rangle = k^{r(N-r)/N^2} \tag{23}$$

for $r = 0, \ldots, N$. This is the result previously conjectured on the basis of series expansions [Eq. (1.20) of Ref. [8]]. We expect all functions to be analytic in the physical ferromagnetically ordered regime (with positive real Boltzmann weights) so our working and results should remain true throughout $0 < k' < 1$. The magnetic critical exponents are $\beta_r = r(N - r)/(2N^2)$.

In [15] we go on to obtain $G_{pq}(r)$ from a Wiener-Hopf factorization of x_p. The result is a special case of the $\tau_2(p')$ free energy. For $r = N$, it and $L_r(t_p)$ can be obtained from (10d). For $N = 3$ we have verified that these results are consistent with previously obtained series expansions [Eqs. (48)–(52) of Ref. [16]].

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