Viscosity in Strongly Interacting Quantum Field Theories from Black Hole Physics

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The ratio of shear viscosity to volume density of entropy can be used to characterize how close a given fluid is to being perfect. Using string theory methods, we show that this ratio is equal to a universal value of $\hbar/4\pi k_B$ for a large class of strongly interacting quantum field theories whose dual description involves black holes in anti–de Sitter space. We provide evidence that this value may serve as a lower bound for a wide class of systems, thus suggesting that black hole horizons are dual to the most ideal fluids.

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Introduction.—It has been known since the discovery of Hawking radiation [1] that black holes are endowed with thermodynamic properties such as entropy and temperature, as first suggested by Bekenstein [2] based on the analogy between black hole physics and equilibrium thermodynamics. In higher-dimensional gravity theories there exist solutions called black branes, which are black holes with translationally invariant horizons [3]. For these solutions, thermodynamics can be extended to *hydrodynamics*—the theory that describes long-wavelength deviations from thermal equilibrium [4]. In addition to thermodynamic properties such as temperature and entropy, black branes possess hydrodynamic characteristics of continuous fluids: viscosity, diffusion constants, etc. From the perspective of the holographic principle [5,6], a black brane corresponds to a certain finite-temperature quantum field theory in fewer number of spacetime dimensions, and the hydrodynamic behavior of a black-brane horizon is identified with the hydrodynamic behavior of the dual theory. For these field theories, in this Letter we show that the ratio of the shear viscosity to the volume density of entropy has a universal value

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13}\,\mathrm{K\,s}. \quad (1)$$

Furthermore, we shall argue that this is the lowest bound on the ratio η/s for a wide class of thermal quantum field theories.

Viscosity and graviton absorption.—Consider a thermal field theory whose dual holographic description involves a D-dimensional black-brane metric of the form

$$\begin{aligned} ds^2 &= g^{(0)}_{MN} dx^M dx^N \\ &= f(\xi)(dx^2 + dy^2) + g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu. \end{aligned} \quad (2)$$

[The $O(2)$ symmetry of the background is required for the existence of the shear hydrodynamic mode in the dual theory, thus making the notion of shear viscosity meaningful.] One can have in mind, as an example, the near-extremal D3-brane in type IIB supergravity, dual to finite-temperature $\mathcal{N} = 4$ supersymmetric SU(N_c) Yang-Mills theory in the limit of large N_c, and large 't Hooft coupling [7–10],

$$\begin{aligned} ds^2 = \frac{r^2}{R^2}\left[-\left(1 - \frac{r_0^4}{r^4}\right)dt^2 + dx^2 + dy^2 + dz^2\right] \\ + \frac{R^2}{r^2(1 - r_0^4/r^4)} dr^2, \end{aligned} \quad (3)$$

but our discussion will be quite general. All black branes have an event horizon [$r = r_0$ for the metric (3)], which is extended along several spatial dimensions [x, y, z in the case of (3)]. The dual field theory is at a finite temperature, equal to the Hawking temperature of the black brane.

The entropy of the dual field theory is equal to the entropy of the black brane, which is proportional to the area of its event horizon,

$$S = \frac{A}{4G}, \quad (4)$$

where G is Newton's constant (we set $\hbar = c = k_B = 1$). For black branes A contains a trivial infinite factor V equal to the spatial volume along directions parallel to the horizon. The entropy density s is equal to $a/(4G)$, where $a = A/V$.

The shear viscosity of the dual theory can be computed from gravity in a number of equivalent approaches [11–13]. Here we use Kubo's formula, which relates viscosity to equilibrium correlation functions. In a rotationally invariant field theory,

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt d\mathbf{x} e^{i\omega t} \langle [T_{xy}(t, \mathbf{x}), T_{xy}(0, \mathbf{0})] \rangle. \quad (5)$$

Here T_{xy} is the xy component of the stress-energy tensor (one can replace T_{xy} by any component of the traceless part of the stress tensor). We shall now relate the right-hand side of (5) to the absorption cross section of low-energy gravitons.

According to the gauge-gravity duality [10], the stress-energy tensor $T_{\mu\nu}$ couples to metric perturbations at the

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boundary. Following Klebanov [14,15], let us consider a graviton of frequency ω, polarized in the xy direction, and propagating perpendicularly to the brane. In the field theory picture, the absorption cross section of the graviton by the brane measures the imaginary part of the retarded Green function of the operator coupled to h_{xy}, i.e., T_{xy},

$$\begin{aligned}\sigma_{\rm abs}(\omega) &= -\frac{2\kappa^2}{\omega}\,{\rm Im}\,G^{\rm R}(\omega)\\ &= \frac{\kappa^2}{\omega}\int dtd\mathbf{x}e^{i\omega t}\langle[T_{xy}(t,\mathbf{x}),T_{xy}(0,\mathbf{0})]\rangle,\end{aligned} \tag{6}$$

where $\kappa = \sqrt{8\pi G}$ appears due to the normalization of the graviton's action. Comparing (5) and (6), we find

$$\eta = \frac{\sigma_{\rm abs}(0)}{2\kappa^2} = \frac{\sigma_{\rm abs}(0)}{16\pi G}. \tag{7}$$

Graviton absorption cross section at low energy.—The absorption cross section $\sigma_{\rm abs}$ is calculable classically by solving the linearized wave equation for h^x_y. We now show that $h^x_y = h_{xy}/f$ obeys the equation for a minimally coupled massless scalar in the background (2). This is similar to cosmological tensor perturbations on a Friedmann-Robertson-Walker spacetime, which obey the equation for a massless scalar field [16].

Consider small perturbations around the metric, $g_{MN} = g^{(0)}_{MN} + h_{MN}$. We assume that the only nonvanishing component of h_{MN} is h_{xy}, and that it does not depend on x and y: $h_{xy} = h_{xy}(\xi)$. This field has spin 2 under the $O(2)$ rotational symmetry in the xy plane, which implies that all other components of h_{MN} can be consistently set to zero [12]. Einstein's equations can be written in the form

$$R_{MN} = T_{MN} - \frac{T}{D-2}g_{MN}, \tag{8}$$

where the stress-energy tensor T_{MN} and its trace T depend on other fields such as the dilaton and various forms supporting the background (2), for example, the fields appearing in the low-energy type II string theory. Again, $O(2)$ xy rotational symmetry implies that all perturbations of matter fields can be set to zero consistently. Thus when M and N are x or y, the right-hand side of Einstein's equations reads ($\alpha, \beta = x, y$)

$$T_{\alpha\beta} - \frac{T}{D-2}g_{\alpha\beta} = -\left(\mathcal{L} + \frac{T^{(0)}}{D-2}\right)(\delta_{\alpha\beta}f + h_{\alpha\beta}), \tag{9}$$

where $\mathcal{L}$ is the Lagrange density of matter fields and $T^{(0)}$ is the trace of the unperturbed stress-energy tensor.

$$T_{MN} = -g_{MN}\mathcal{L} + \cdots, \tag{10}$$

where $\mathcal{L}$ represents the Lagrangian density of the fields and dots denote terms of second and higher orders in the perturbation h_{xy}. Substituting the unperturbed metric (2) into Einstein's equations, one finds

$$\frac{1}{2}\left[\frac{\Box f}{f} - \frac{(\partial f)^2}{f^2}\right] = \mathcal{L} + \frac{T^{(0)}}{D-2}. \tag{11}$$

Expanding Einstein's equations to linear order in h_{xy}, one has

$$\begin{aligned}R_{xy} &= -\frac{1}{2}\Box h_{xy} + \frac{1}{f}\partial^\mu f\partial_\mu h_{xy} - \frac{(\partial f)^2}{2f^2}h_{xy}\\ &= -\left(\mathcal{L} + \frac{T^{(0)}}{D-2}\right)h_{xy}.\end{aligned} \tag{12}$$

Combining Eqs. (11) and (12), we obtain an equation for h_{xy}:

$$\Box h_{xy} - 2\frac{\partial^\mu f}{f}\partial_\mu h_{xy} + 2\frac{(\partial f)^2}{f^2}h_{xy} - \frac{\Box f}{f}h_{xy} = 0. \tag{13}$$

Changing the variable from h_{xy} to $h^x_y = h_{xy}/f$, one can see that h^x_y indeed satisfies the equation for a minimally coupled massless scalar: $\Box h^x_y = 0$. The absorption cross section of a graviton is therefore the same as that of the scalar.

The absorption cross section for the scalar is constrained by a theorem [17,18], which states that in the low-frequency limit $\omega \to 0$ this cross section is equal to the area of the horizon, $\sigma_{\rm abs} = a$. Since $s = a/4G$, one immediately finds that

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}, \tag{14}$$

where $\hbar$ and k_B are now restored. This shows that the ratio η/s does not depend on the concrete form of the metric within the assumptions of Refs. [17,18]. Indeed, this ratio is the same for Dp- ([11,13]), M2-, and M5- ([19]) branes and for deformations of the D3 metric [13,20]. This fact is very surprising, given that the corresponding dual field theories are very different. We do not have an explanation for the constancy of η/s in these theories based on field-theoretical arguments alone.

A viscosity bound conjecture.—Most quantum field theories do not have simple gravity duals. Is our result relevant in a broader setting? We speculate that the ratio η/s has a lower bound

$$\frac{\eta}{s} \geqslant \frac{\hbar}{4\pi k_B} \tag{15}$$

for all relativistic quantum field theories at finite temperature and zero chemical potential. The inequality is saturated by theories with gravity duals.

One argument supporting the bound (15) comes from the Heisenberg uncertainty principle. The viscosity of a plasma is proportional to $\epsilon\tau_{\rm mft}$, where ϵ is the energy density and $\tau_{\rm mft}$ is the typical mean free time of a quasiparticle. The entropy density, on the other hand, is proportional to the density of quasiparticles, $s \sim k_B n$. Therefore, $\eta/s \sim k_B^{-1}\tau_{\rm mft}\epsilon/n$. Now ϵ/n is the average energy per particle. According to the uncertainty principle, the product of the energy of a quasiparticle ϵ/n and its mean free time $\tau_{\rm mft}$ cannot be smaller than $\hbar$, otherwise the quasiparticle concept does not make sense. Therefore we obtain, from the uncertainty principle alone, that $\eta/s \gtrsim \hbar/k_B$, which is (15) without the numerical coefficient of $1/(4\pi)$. We also conclude that η/s is much larger than $\hbar/k_B$ in weakly coupled theories (where the mean free time is large).

Another piece of evidence supporting the bound (15) comes from a recent calculation [21] of η/s in the $\mathcal{N} = 4$ supersymmetric SU(N_c) Yang-Mills theories in the regime of infinite N_c and large, but finite, 't Hooft coupling g^2N_c. The first correction in inverse powers of g^2N_c corresponds to the first string theory correction to Einstein's gravity. The result reads

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\left[1 + \frac{135\zeta(3)}{8(2g^2N_c)^{3/2}} + \cdots\right], \qquad (16)$$

where $\zeta(3) \approx 1.2020569\ldots$ is Apéry's constant. The correction is positive, in accordance with (15). It is natural to assume that η/s is larger than the bound for all values of the 't Hooft coupling (Fig. 1).

The bound (15), in contrast to the entropy bound [22] and Bekenstein's bound [23], does not involve the speed of light c and hence is nontrivial when applied to nonrelativistic systems. However, the range of applicability of (15) to nonrelativistic systems is less certain. On the one hand, by subdividing the molecules of a gas to an ever-increasing number of nonidentical species one can increase the entropy density (by adding the Gibbs mixing entropy) without substantially affecting the viscosity. On the other hand, the conjectured bound is far below the ratio of η/s in any laboratory liquid. For water under normal conditions, η/s is 380 times larger than $\hbar/(4\pi k_B)$. Using standard tables [24,25] one can find η/s for many liquids and gases at different temperatures and pressures. Figure 2 shows temperature dependence of η/s, normalized by $\hbar/(4\pi k_B)$, for a few substances at different pressures. It is clear that the viscosity bound is well satisfied for these substances. Liquid helium reaches the smallest value of η/s, but this value still exceeds the bound by a factor of about 9. We speculate that the bound (15) is valid at least for a single-component nonrelativistic gas of particles with spin 0 or 1/2.

Discussion.—It is important to avoid some common misconceptions which at first sight seem to invalidate the viscosity bound. Somewhat counterintuitively, a near-ideal gas has a very large viscosity due to the large mean free path. Likewise, superfluids have finite and measurable shear viscosity associated with the normal component, according to Landau's two-component theory.

The bound (15) is most useful for strongly interacting systems where reliable theoretical estimates of the viscosity are not available. One of such systems is the quark-gluon plasma (QGP) created in heavy ion collisions which behaves in many respects as a droplet of a liquid. There are experimental hints that the viscosity of the QGP at temperatures achieved by the Relativistic Heavy Ion Collider is surprisingly small, possibly close to saturating the viscosity bound [26]. Another possible application of the viscosity bound is trapped atomic gases. By using the Feshbach resonance, strongly interacting Fermi gases of atoms have been created recently. These gases have been observed to behave hydrodynamically [27] and should have finite shear viscosity at nonzero temperature. It would

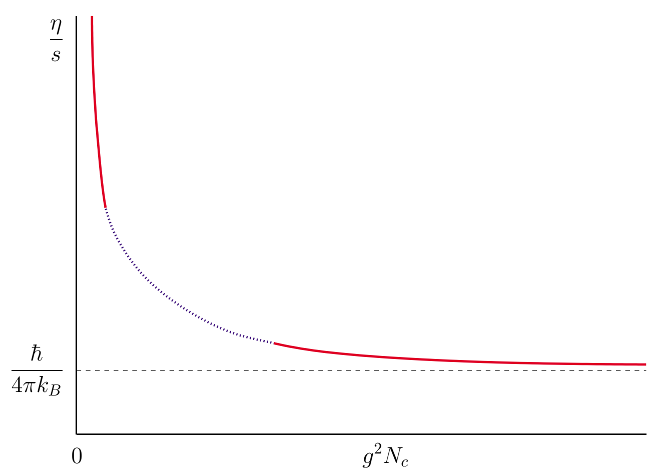

FIG. 1 (color online). The dependence of the ratio η/s on the 't Hooft coupling g^2N_c in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The ratio diverges in the limit $g^2N_c \to 0$ and approaches $\hbar/4\pi k_B$ from above as $g^2N_c \to \infty$. The ratio is unknown in the regime of intermediate 't Hooft coupling.

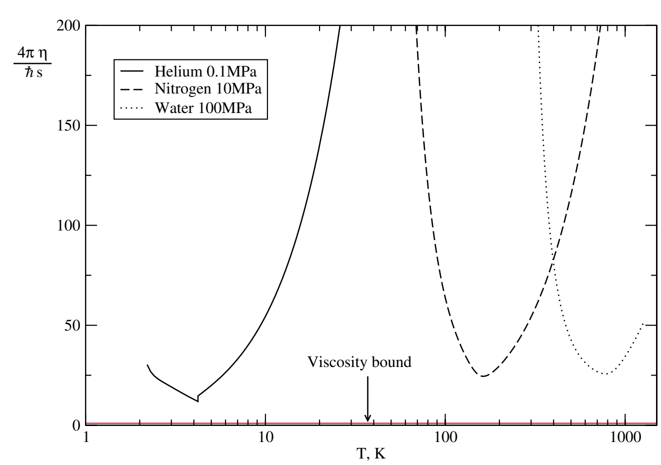

FIG. 2 (color online). The viscosity-entropy ratio for some common substances: helium, nitrogen and water. The ratio is always substantially larger than its value in theories with gravity duals, represented by the horizontal line marked "viscosity bound."

be very interesting to test experimentally whether the shear viscosity of these gases satisfies the conjectured bound.

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