Universal Power-Law Decay of the Impulse Energy in Granular Protectors

Jongbae Hong
School of Physics, Seoul National University, Seoul 151-747, Korea
(Received 19 November 2004; published 14 March 2005)

Protecting a big impulse from outside is one of the important issues of our everyday life. A granular medium is often used as a protecting material. The impulse inside a granular medium is a solitary wave which may be confined temporarily to a particular region of the medium, which we call the granular container that plays the role of the protector. We find a universal power-law behavior in time for the leakage of the impulse energy confined inside various granular containers.

DOI: 10.1103/PhysRevLett.94.108001 PACS numbers: 45.70.−n, 02.70.Ns, 43.25.+y, 46.40.Cd

There could be various kinds of disastrous external impacts, such as an earthquake, bomb explosion, automobile collision, and so on. People hope to protect something important from these mechanical impacts. One possible way of effective protection is to confine the impulse to a specific region. But it is practically impossible to confine the impulse in a certain region perfectly and permanently. However, one may construct an effective protector that confines a strong impulse inside it for a short time and makes the strong impulse into many weak impulses, then releases them outside the protector little by little. Confining a big impulse into a specific region, disintegrating it into many weak impulses, and releasing it with time lag could be an effective way to protect against a strong external impact. This kind of protection mechanism could be realized in a specially prepared granular medium.

Granular matter is ubiquitous around us and has already been used as a protector in our everyday life. However, fundamental research for the granular protector has not been done much because of its complication and nonlinear nature [1]. It has been proved analytically [2,3] and numerically [4] that the propagating mode in the granular chain with power-law type contact force, i.e., $F \propto \delta^p$, where δ is the squeezed distance between neighboring grains, is a solitary wave. The solitary wave in a granular chain with Hertzian contact force, $p = 3/2$, can be described by a soliton in a continuum limit [5]. Some of soliton properties predicted by theory [5] have been demonstrated by experiment [6]. This soliton, or the solitary wave, shows interesting anomalous features of propagation when it passes an interface of a granular medium composed of different granules, which are discriminated by mass, size, and elastic property. In this Letter, we consider spherical granules of the same size. Therefore, the ratio of mass to elastic property is the only parameter that discriminates granules. For simplicity, we assume the same elastic property, and the granules are discriminated by their masses only.

A known, but not widely known, anomalous feature of the wave propagation in the granular chain is the total transmission of a solitary wave along with impulse disintegration when it passes the interface from the region of heavy granules to that of light granules [7,8]. The number of disintegrated solitary waves depends on the strength of precompression. But it is about the mass ratio of heavy to light granules when the precompression is weak. An example of disintegration of a big solitary wave into smaller solitary waves is shown in Fig. 1. The leading solitary wave after transmission is highest and fastest, i.e., the higher, the faster. The followers are lower and slower gradually as shown in Fig. 1 [7,9]. No impulse disintegration and no total transmission occur when a solitary wave propagates from the region of light granules to that of heavy granules. This anomalous behavior of total transmission in an inhomogeneous granular chain provides an idea of containing incident impulse inside a granular container. Furthermore, the property of disintegration of a big solitary wave into many smaller ones provides an idea of effective protection by reducing a strong impulse into many weak ones inside the granular container.

We propose a standard type of effective granular container that is composed of a series of granular sections of different contact force and mass as shown in Fig. 2(a). Three sections of linear medium are added for the walls of the container. The scheme of effective protection is as follows. An impulse reaching one end of the protector

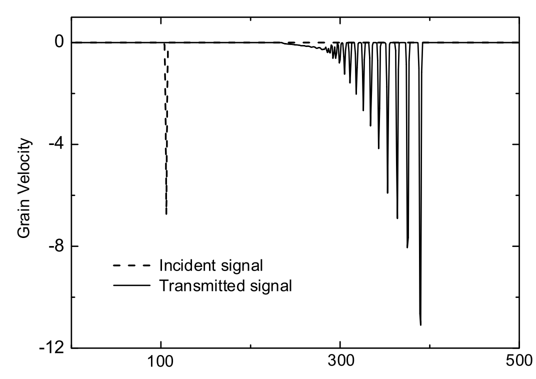

FIG. 1. Snapshots of a solitary wave before and after passing an interface between two granular media of mass ratio 10. The mass of a granule on the left side is 10 times larger than that on the right side and no precompression is applied.

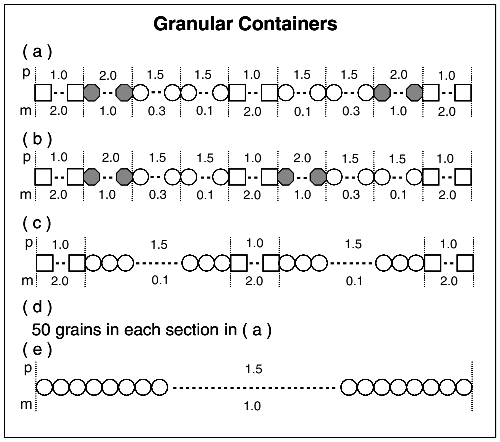

FIG. 2. Schematic diagrams of granular containers. Circles, grey octagons, and squares represent the granule of $p = 1.5$, $p = 2$, and $p = 1$, respectively. The mass of the granule is denoted by m. (a) is the standard type.

proceeds up to the central section without reflection. But the impulse disintegrates into small solitary waves when it passes through each interface, because the mass of granules in each section decreases. This impulse disintegration lasts until the leading solitary wave reaches the edge of the central section. Then, both transmission and reflection occur simultaneously at each interface when the solitary wave proceeds from the central section to the side walls of the granular container. The process of propagation inside the granular container of Fig. 2(a) allows a time lag when the disintegrated impulses leave the granular container. Several variations, Figs. 2(b)–2(e), from the standard type will be studied to compare the effectiveness of protection.

The granular container as a practical protector must be three-dimensional. Usual three-dimensional granular systems have complicated distribution of force chains through which impulses may transmit. The nature of propagation of an impulse inside a three-dimensional granular system is not as simple as one shown in a one-dimensional chain. Therefore, one must construct an artificial three-dimensional protector by stacking granular chains to retain the features mentioned above.

To study the time dependence of energy leakage from the container, we focus on the motion of grains in a horizontal granular chain with interfaces. Simulating the motion of grains and studying propagating characteristics of solitary waves has been done in many previous works [4,5,7–11]. The equation of motion of a grain is determined by the contact force between neighboring grains. For spherical grains, the contact force follows Hertz's law [7,12],

$$F = \frac{2E}{3(1-\nu^2)}\sqrt{\frac{R_1 R_2}{R_1 + R_2}}[(R_1 + R_2) - (x_2 - x_1)]^{3/2}, \quad (1)$$

where R_i is the grain radius, E is the Young's modulus of the material, ν is Poisson's ratio, and x_i is the coordinate of the center of the ith granule with $x_2 > x_1$. We consider in this Letter a general power-type contact force. Then the equation of motion of a grain under precompression δ_0 is written as

$$m\ddot{u}_n = \eta[\{\delta_0 - (u_n - u_{n-1})\}^p - \{\delta_0 - (u_{n+1} - u_n)\}^p], \quad (2)$$

where u_i is the displacement of the ith grain from its initial equilibrium position due to external perturbation, m is the mass of the grain, and η is the elastic constant of the grain depending on its radius, Young's modulus, and Poisson's ratio [6,7]. The only parameter governing Eq. (1) for a given precompression is the ratio η/m. We fix η for all grains and vary masses to discriminate different granules. Therefore, in this Letter, heavy and light granules mean small and large η/m, respectively. In our numerical simulation, we set $\delta_0 = 0$ under which a solitary wave is also created [2,3].

We choose Hertzian contact for most granules. But another nonlinear contact, $p = 2$, which corresponds to grains with irregular surfaces such as sand [13], is used in some cases. To perform numerical simulation for Eq. (2), we choose 10^{-5} m, 2.36×10^{-5} kg, and 1.0102×10^{-3} s as the units of distance, mass, and time, respectively. The integration time step has been kept at 1.25×10^{-5}, which corresponds to 1.26×10^{-8} s in reality. Since we take a snapshot in every 10^4 integration time step, the real time interval of the snapshot is 0.126 ms.

The grain diameter used in this work is 100, i.e., 1 mm, and $\eta = 5657$. For the case of the stainless steel ball [6], $\eta = 2618 \times 10^6 (\mathrm{N/m^{3/2}})$ for a diameter of 1 mm. To make our simulation data realistic for a stainless steel ball of diameter 1 mm, time rescaling by $(5657/2618)^{1/2} \times 10^{-3} t$ is required in Eq. (2). Thus, the time interval of the snapshot for a stainless steel ball of diameter 10^{-3} m becomes 0.40 μs.

Figure 3 shows energy leaking in the form of small solitary waves leaving out of the granular container of Fig. 2(a) with 20 granules in each section. But the mass of a granule in three $p = 1$ sections has been changed into $m = 10$ instead of $m = 2$. To see the leaking solitary waves from the container, we put heavy Hertzian grains of $m = 100$ in either side of the container. Then we apply an initial impulse to the right end of the granular chain using a grain of mass 100 with velocity 10 in our program units, which corresponds to 5.4×10^6 m/s for the steel ball mentioned above. Figure 3 is the 280th snapshot of grain velocity which corresponds 0.112 ms after the collision for the steel ball mentioned above.

It is interesting to see the energies of small solitary waves leaving out of the granular container. The energy of a solitary wave is the sum of the mechanical energies of grains participating in solitary waves. The number of

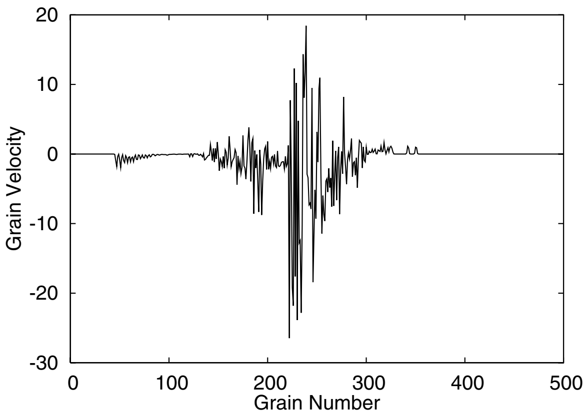

FIG. 3. The 280th snapshot of energy leakage for the granular container of Fig. 2(a). The container ranges from 121 to 300 in grain number, and granules of $m = 100$ are placed outside the container.

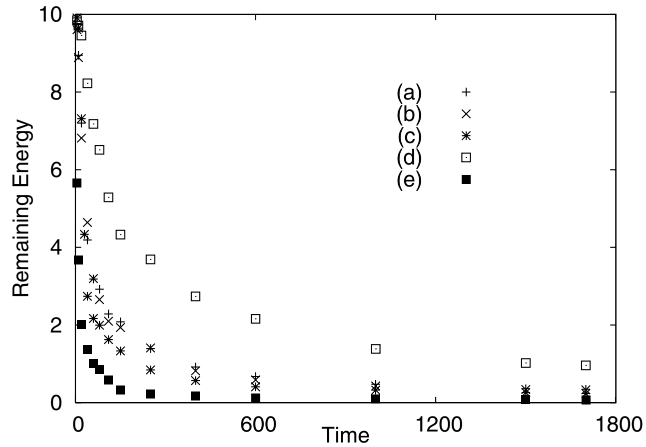

FIG. 4. Impulse energies remaining inside the granular container of the types in Fig. 2. Time scale denotes the number of snapshots.

grains composing a solitary wave is approximately 5 for any height of solitary wave. This is a unique property of the Nesterenko soliton [5] appearing in a granular chain. The energies of leading solitary waves leaving the granular container appearing at the right and left ends of the snapshot of Fig. 3 are, respectively, 3.3% and 7.7% of the energy of the incident solitary wave. One can see that a strong initial impulse incident on the granular container is broken into small solitary waves whose individual energy is less than 10% of the original energy when they leave the granular container. Therefore, designing a specific granular container is a way of constructing an effective protector.

It is natural for us to pay attention to the remaining energy inside the granular container as time elapses. A fascinating universal behavior is discovered in the energy-leaking process from the granular container. Figure 4 shows the plots of the energy remaining inside the granular container versus elapsed time for those variations of Figs. 2(b)–2(e) alongside the standard case of Fig. 2(a). Figure 2(b) has the same structure as the standard one, but the sequence of sections in the right half is reversed. Figure 2(c) has the same 60 granules of mass 0.1 and $p = 1.5$ on each side of the central section. Figure 2(d) has the same structure as the standard type, but the number of granules in each section is 50 instead of 20. Finally, we introduce the simplest granular container composed of the same 180 nonlinear granules of $p = 1.5$ and $m = 1.0$ without linear walls as shown in Fig. 2(e). As one might expect, increasing the number of granules in each section, such as Fig. 2(d), will cause slower leaking, while using the same mass of granules, such as Fig. 2(c), will cause faster leaking. The reason for slower or faster leaking of energy surely stems from the time taken for a solitary wave to pass from one edge of the container to the other edge. To see this more clearly, we add the one shown in Fig. 2(e) for comparison, which takes the shortest elapsed time passing from one end to the other and results in the fastest leakage.

A remarkable universality behavior of energy leaking from different types of granular containers is seen if we replot the same data of Fig. 4 on a log-log scale in Fig. 5 through time rescaling. We rescale time by $2t$, $0.22t$, and $9t$ for (c), (d), and (e) of Fig. 2, respectively. Falling on a single line after proper scaling stresses that the nature of energy leaking is universal, i.e., the same power-law exponent in time. In other words, the essential time dependence of energy leaking from the granular container is independent of its construction. The inverse scaling factors qualitatively describes relative time scale for a solitary wave to stay inside a granular container.

The universal power-law behavior in time for the energy leaking from the granular container is a new discovery. Therefore, understanding of the universal power-law be-

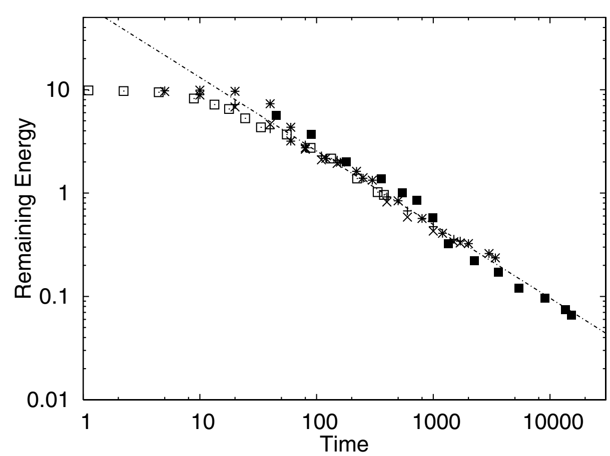

FIG. 5. The log-log plots of the data of Fig. 4 expressed by time rescaling. Times are rescaled by multiplying 2, (2/9), and 9 for the data designated by stars, open squares, and solid squares, respectively.

havior is an interesting subject to pursue. One can understand that energy leaking from the container occurs when a solitary wave undergoes both transmission and reflection at the interface in the edge of the container. This happens when a solitary wave passes through the interface from a light- to a heavy-granular medium. In this process of energy leaking, the solitary wave decreases its height and speed. Therefore, the energy remaining inside the granular container depends on the number of reflections at both interfaces of the container walls, and the number of reflections is proportional to the speed of the solitary wave and inversely proportional to the length of the container. One may construct an equation of motion for the energy remaining inside the granular container based on this analysis.

If there were no reduction in the speed of the solitary wave after reflection, one may easily guess that the remaining energy would decay exponentially. That is, the equation of motion for the remaining energy is written as $E_R(t + \Delta t) - E_R(t) = -k\Delta t E_R(t)$, where k is the inverse of the time constant. The energy decay in this study, however, is not this simple, because the speed of the solitary wave is not constant but decreases after reflection. Therefore, the constant k is not constant any more but time dependent. According to the analysis above, k must be replaced by $\langle N_r(t)\rangle$ that is the averaged number of reflections per unit time at time t. Thus, the change in the energy remaining inside the granular container is written as $E_R(t + \Delta t) - E_R(t) \propto -\langle N_r(t)\rangle \Delta t E_R(t)$. Since the speed of the solitary wave decreases after every reflection, $\langle N_r(t)\rangle$ must decrease also. One can infer the time-dependent behavior of $\langle N_r(t)\rangle$ as $\langle N_r(t)\rangle \propto 1/t$ from the data of Fig. 5. Then, we finally set up the equation of motion for the change in the remaining energy as follows:

$$E_R(t + \Delta t) - E_R(t) = -\gamma(\Delta t/t)E_R(t), \tag{3}$$

which gives the solution

$$E_R(t) = At^{-\gamma}, \tag{4}$$

where γ is a universal dimensionless constant and the constant A depends on the structure of the granular container, such as the length of the container and the arrangement of the granules. Figure 5 gives $\gamma = 0.7055$ and $A = 65.31 \times \alpha^{\gamma}$, where α is the time scaling factor. We choose $\alpha = 1$ for our standard type of Fig. 2(a).

In conclusion, we find an interesting universal behavior in energy leaking from a granular container. Various types of granular container show the same power-law type energy-leaking behavior in time. We understand that this power-law behavior stems from the decreasing of the speed of a solitary wave after the reflection accompanying transmission, through which mechanism the energy of impulse leaks from the container. Reduction of the speed of the solitary wave in time causes a reduction in the average number of reflections per unit time as time elapses. The universal power-law behavior in time of the energy remaining inside the granular container implies that the rate of energy leakage becomes slower due to the slowing down of solitary waves after reflection accompanying transmission and it is proportional to $(1/t)$. A big solitary wave produced by a strong external impulse is broken into many small solitary waves when it passes an interface from a small η/m- to a large η/m-granular medium. Therefore, a granular container having multiple interfaces with an appropriate alignment of η/m may play a role as an effective protector for an external impulse.

The mechanism of protection is holding the energy carried by solitary waves inside the container and releasing the energy little by little in the form of separate solitary waves as shown in Fig. 3. The phenomena of impulse confinement and disintegration appearing in a granular chain discussed here may also appear in other systems, such as electromagnetic devices and biomolecular chains, if a power-law type nonlinear interaction exists between the elements of the system. Application to these systems would be more interesting.

The author thanks J. H. Bang and E. J. Hong for numerical work. This work was supported by the Korea Research Foundation Grant No. (KRF-2002-015-CP0131).

[1] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
[2] R. S. MacKay, Phys. Lett. A **251**, 191 (1999).
[3] J.-Y. Ji and J. Hong, Phys. Lett. A **260**, 60 (1999).
[4] R. S. Sinkovits and S. Sen, Phys. Rev. Lett. **74**, 2686 (1995); S. Sen, M. Manciu, and J. D. Wright, Phys. Rev. E **57**, 2386 (1998).
[5] V. F. Nesterenko, J. Appl. Mech. Tech. Phys. **5**, 733 (1983); V. F. Nesterenko, J. Phys. IV (France) **55**, C8-729 (1994).
[6] C. Coste, E. Falcon, and S. Fauve, Phys. Rev. E **56**, 6104 (1997).
[7] V. F. Nesterenko, *Dynamics of Heterogeneous Materials* (Springer, New York, 2001), Chap. 1.
[8] J. Hong and A. Xu, Appl. Phys. Lett. **81**, 4868 (2002).
[9] E. Hascoët and H. J. Herrmann, Eur. Phys. J. B **14**, 183 (2000).
[10] A. Chatterjee, Phys. Rev. E **59**, 5912 (1999).
[11] J. Hong, H. Kim, and J.-P. Hwang, Phys. Rev. E **61**, 964 (2000); J. Hong and Aiguo Xu, Phys. Rev. E **63**, 061310 (2001).
[12] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1970), p. 30; H. Hertz, Journal fur die reine und angewandte Mathematik **92**, 156 (1881).
[13] J. D. Goddard, Proc. R. Soc. London A **430**, 105 (1990).