Dissolving $\mathcal{N} = 4$ Loop Amplitudes into QCD Tree Amplitudes

Radu Roiban,[1] Marcus Spradlin,[2] and Anastasia Volovich[2]
[1]Department of Physics, Princeton University, Princeton, New Jersey 08544, USA
[2]Kavli Institute for Theoretical Physics, Santa Barbara, California 93106, USA
(Received 23 December 2004; published 16 March 2005)

We use the infrared consistency of one-loop amplitudes in $\mathcal{N} = 4$ Yang-Mills theory to derive a compact analytic formula for a tree-level next-to-next-to-maximal helicity-violating gluon scattering amplitude in QCD, the first such formula known. We argue that the infrared conditions, coupled with recent advances in calculating one-loop box coefficients, can give a new tool for computing tree-level amplitudes in general. Our calculation suggests that many amplitudes have a structure which is even simpler than that revealed so far by current twistor-space constructions.

DOI: 10.1103/PhysRevLett.94.102002 PACS numbers: 12.38.Bx, 11.15.Bt, 11.55.Bq

Introduction.—Gluon scattering amplitudes in Yang-Mills theory have many remarkable properties. Witten discovered that they are localized on curves in twistor space [1] and proposed an interpretation for this fact in terms of a twistor string theory. There has been great success in calculating tree-level [1–7] amplitudes using twistor-inspired technology, including the construction of a one-line (though implicit) formula for the complete tree-level S matrix [4].

Twistor string theory led in [5] to the development of a set of diagrammatic rules for the calculation of tree-level amplitudes which are far simpler than standard Feynman diagrams. Nevertheless, there is reason to believe that the formulas generated by the Cachazo-Svrček-Witten (CSW) rules grow in complexity much more quickly than the underlying amplitudes do. In this Letter we report a compact analytic representation for the simplest next-to-next-to-maximal helicity-violating (NNMHV) amplitude [8]

$$A(1^-,2^-,3^-,4^-,5^+,6^+,7^+,8^+) = \frac{\langle 1|k_2^{[3]}|5]^3}{t_2^{[4]}[23][34][45]\langle 67\rangle\langle 78\rangle\langle 81\rangle[2|k_3^{[3]}|6\rangle} - \frac{1}{t_7^{[3]}\langle 78\rangle\langle 81\rangle[3|k_4^{[2]}|6\rangle\langle 7|k_8^{[2]}|2]}\left[\frac{\langle 1|k_7^{[2]}k_3^{[4]}k_3^{[2]}|5]^3}{t_3^{[3]}t_3^{[4]}[34][45][2|k_3^{[3]}|6\rangle} + \frac{\langle 1|k_7^{[2]}k_5^{[2]}|4\rangle^3}{t_4^{[3]}[23]\langle 45\rangle\langle 56\rangle}\right] + \overline{\text{flip}}, \tag{1}$$

which otherwise would require summing 44 CSW diagrams to write down [10]. Previously, no such compact expression was known for any NNMHV amplitude. The "$+\overline{\text{flip}}$" denotes that this full expression should be added to its image under the operation which relabels $i \to 9 - i$ and simultaneously exchanges $\langle\rangle$ and $[]$.

Gluon scattering amplitudes are obviously of phenomenological interest as they are the basic building block for computing QCD backgrounds to jet production. From a theoretical standpoint, perhaps the most interesting fact about the formula (1) is that such a compact representation of this amplitude exists at all. The existence of this formula suggests that there is hidden structure and simplicity underlying tree amplitudes even beyond what the CSW rules expose [11]. Moreover, the formula (1) is much simpler than we had any right to expect, given the procedure used to compute it. Rather, (1) hinges upon the miraculous cancellation of a large number of terms, which at the moment we cannot explain. The lesson we would like to draw from this exercise is that there still is a lot to learn about the structure of Yang-Mills amplitudes, even at tree level, and, in particular, that CSW is not the end of the story [13].

Recently there has also been much progress on the computation of one-loop amplitudes [9,12,14–28]. In particular, a prescription has recently been developed [27] for calculating the coefficient of any one-loop box function in the $\mathcal{N} = 4$ theory, based on the notion of generalized cuts [9,29–31] (in this case, quadruple cuts). One-loop amplitudes have infrared (IR) singularities, and the leading singularities are proportional to tree amplitudes in a way that we review below. Enforcing this fact may provide, as pointed out in [9], new and more compact expressions for tree-level scattering amplitudes. We derived the tree-level formula (1) by computing several box coefficients appearing in the one-loop amplitude corresponding to (1) and then reading off its IR singularity. In the next section we review the necessary ingredients and discuss a couple of strategies for maximizing the efficiency of this kind of computation.

Trees from loops.—The set of dimensionally regularized integrals which can appear at one loop has been completely classified. Any n-gluon partial amplitude in $(4 - 2\epsilon)$-dimensional $\mathcal{N} = 4$ super-Yang-Mills theory can be expressed in terms of $\binom{n}{4}$ box functions $B_n(i, j, k, l)$

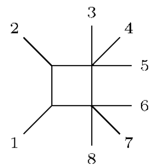
 © 2005 The American Physical Society

$$A_{n;1\text{loop}} = ic_\Gamma(\mu^2)^\epsilon \sum c_{n;ijkl} B_n(i,j,k,l),$$
$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{2}$$

where μ is a renormalization scale. The box functions are completely known (but complicated) functions of the momenta p_i of the n gluons. Therefore the decomposition (2) reduces the problem of calculating any one-loop amplitude to the problem of calculating the (relatively much simpler) box coefficients $c_{n;ijkl}$, which can depend on the momenta as well as the helicities of the n gluons.

The box functions are typically transcendental functions of the momenta (involving logarithms and dilogarithms). They are most conveniently labeled [12] by a set of distinct, ordered indices i, j, k, l chosen from the set $\{1, \ldots, n\}$. For example,

$$B_8(1,2,3,6) =$$



$$= -\frac{1}{2\epsilon^2}\left[(-t_1^{[2]})^{-\epsilon} + 2(-t_2^{[4]})^{-\epsilon} - (-t_3^{[3]})^{-\epsilon} - (-t_6^{[3]})^{-\epsilon}\right] + \mathrm{Li}_2\left(1 - \frac{t_3^{[3]}}{t_2^{[4]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_6^{[3]}}{t_2^{[4]}}\right) + \frac{1}{2}\ln^2\left(\frac{t_1^{[2]}}{t_2^{[4]}}\right) + \mathcal{O}(\epsilon). \tag{3}$$

The indices indicate the four external legs which immediately follow (in a clockwise sense, conventionally) the four internal propagators in the corresponding box integral. Explicit formulas for all box functions can be found in any standard reference, and we will not repeat them all here but rather provide (3) as an illustrative example for what follows.

Each box function contains $\mathcal{O}(1/\epsilon)$ terms of the form $-\ln(-t_i^{[r]})/\epsilon$ for various i and r, with coefficients 0, ± 1, or $\pm\frac{1}{2}$. Meanwhile, the $\mathcal{O}(1/\epsilon)$ IR singularity in any one-loop amplitude is known on general grounds to be [32–34]

$$A_{n;1\text{loop}}\big|_{\epsilon\text{pole}} = -\frac{c_\Gamma}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-t_i^{[2]}}\right)^\epsilon A_n^{\text{tree}}. \tag{4}$$

One constraint on the box coefficients is that they conspire in such a way that the IR divergences in the box functions combine together so that this equation is satisfied. A set of $n(n-3)/2$ linear equations, n of which involve the tree amplitude, can be deduced by equating (2) to (4) and reading off the coefficients of the various $-\ln(-t_i^{[r]})/\epsilon$ poles.

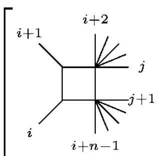
It is likely that in many cases, the IR equations implied by (4) allow one to write down explicit formulas for tree amplitudes where none was previously known, or to write down more compact formulas for previously known amplitudes. An example of the latter is the remarkable formula

$$A(1^-,2^-,3^-,4^+,5^+,6^+) = \frac{\langle 1|k_2^{[2]}|4]^3}{t_2^{[3]}[23][34]\langle 56\rangle\langle 61\rangle[2|k_3^{[2]}|5\rangle} + \frac{[6|k_1^{[2]}|3\rangle^3}{t_6^{[3]}[21][16]\langle 54\rangle\langle 43\rangle[2|k_6^{[2]}|5\rangle}, \tag{5}$$

which follows from the collinear limit of a three-term representation of the 7-particle amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+,7^+)$ found in [9] using the IR equations for $n=7$.

A particularly compact formula which follows from (4) is

$$A^{\text{tree}}(1,\ldots,n) = \frac{1}{2}\sum_{j=i+2}^{i+n-2} \text{coefficient}\left[\quad\right], \tag{6}$$



for any i. This formula expresses an arbitrary tree-level amplitude as a sum of $n-3$ box coefficients of the corresponding one-loop amplitude. For $n=8$ we have verified that no linear combination of the IR equations allows one to extract the tree amplitude from fewer than $n-3=5$ coefficients. We conjecture that this remains true for any n, so that (6) is in a sense the most efficient linear combination of IR equations possible. Of course, not all box coefficients are equally simple to compute, so in practice it may be worthwhile to consider a linear combination of IR equations which has a larger number of terms which are however individually simpler to compute. We will encounter an example of this in the next section.

The $A(1^-,2^-,3^-,4^-,5^+,6^+,7^+,8^+)$ *amplitude.*—In this section we investigate the tree-level eight-gluon amplitude

$$A_8 \equiv A(1^-,2^-,3^-,4^-,5^+,6^+,7^+,8^+), \tag{7}$$

for which no compact analytic formula was previously known. As a preliminary remark we note that A_8 possesses two commuting $\mathbb{Z}_2$ symmetries. One is the symmetry under the flip operation defined by

$$F[X] = X(1 \leftrightarrow 4, 2 \leftrightarrow 3, 5 \leftrightarrow 8, 6 \leftrightarrow 7). \tag{8}$$

The other is a conjugation symmetry combined with a relabeling of the indices, which we will denote by

$$G[X] = \overline{X}(1 \leftrightarrow 5, 2 \leftrightarrow 6, 3 \leftrightarrow 7, 4 \leftrightarrow 8). \tag{9}$$

The bar over the X denotes the reversal of the helicities of the inner products, i.e., $\langle\rangle \leftrightarrow []$.

In order to calculate A_8 we use the IR equations introduced in the previous section, combined with the new technology of [27] for calculating one-loop box coefficients using quadruple cuts. In this approach, the coefficient of a box function is given by the product of the four tree amplitudes sitting at the corners of the box. This construction implies that the complexity of a box coefficient is indicated by the complexity of the tree amplitudes sitting at the corners of the box. Moreover, when coupled with the IR equations, it implies that various tree amplitudes satisfy recursion relations relating n-particle amplitudes to amplitudes with fewer particles. In general these relations are quartic, although we see that the choice (6) actually renders them quadratic since only two of the four corners involve a nontrivial lower-point tree amplitude.

In particular, the simplest box coefficients are those which have a maximally-helicity-violating (MHV) amplitude on each corner, which is guaranteed to happen when each corner has fewer than four external legs. Let us refer to these as "$(\mathrm{MHV})^4$ boxes". For $n = 8$, it turns out that there is an almost unique [35] linear combination of IR equations which expresses the desired tree amplitude A_8 in terms of the coefficients of $(\mathrm{MHV})^4$ boxes only:

$$4A_8 = X + F[X] + G[X] + F[G[X]], \tag{10}$$

where

$$X = X_1 = c_{1236} + c_{1346} + c_{1347} + \frac{1}{2}c_{2347} + c_{2367} + c_{2368}. \tag{11}$$

Although it is intriguing that it is possible to express the tree amplitude A_8 in a manifestly symmetric manner using only $(\mathrm{MHV})^4$ boxes, it is not clear that (10) would provide the most compact representation of A_8.

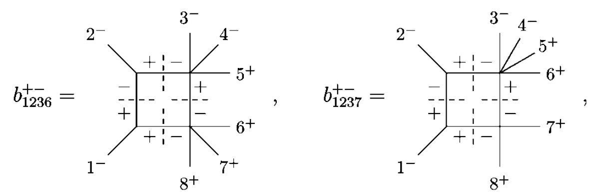
An alternate representation of A_8 is given by the same formula (10), but with

$$X = X_2 = c_{1236} + c_{1237} + c_{1246} + c_{1247} + \frac{1}{2}c_{1258}, \tag{12}$$

where we omitted a term c_{1256}, which is easily seen to be zero by analyzing the holomorphic anomaly of the $t_1^{[4]}$ cut. We have worked out all of the coefficients appearing in (12), and the purpose of this Letter is to point out a most unexpected surprise: almost all of the terms appearing on the right-hand side of (12) cancel [after summing over the $\mathbb{Z}_2 \times \mathbb{Z}_2$ images in (12)]. The only terms remaining are

$$X_2 = -\frac{2}{t_1^{[2]}t_2^{[4]}}b_{1236}^{+-} - \frac{2}{t_1^{[2]}t_7^{[3]}}b_{1237}^{+-}, \tag{13}$$

where the b's are partial contributions to the 1236 and 1237 integral coefficients. In particular, they denote the following quadruple cuts:



(14)

which we computed using the methods of [27], giving

$$b_{1236}^{+-} = \frac{t_1^{[2]}[5|k_6^{[3]}|1\rangle^3}{[23][34][45]\langle 67\rangle\langle 78\rangle\langle 81\rangle[2|k_3^{[3]}|6\rangle},$$

$$b_{1237}^{+-} = \frac{t_1^{[2]}}{\langle 78\rangle\langle 81\rangle[3|k_4^{[2]}|6\rangle\langle 7|k_8^{[2]}|2]} \times \left[\frac{\langle 1|k_7^{[2]}k_3^{[4]}k_3^{[2]}|5]^3}{t_3^{[3]}t_3^{[4]}[34][45][2|k_3^{[3]}|6\rangle} + \frac{\langle 1|k_7^{[2]}k_5^{[2]}|4\rangle^3}{t_4^{[3]}[23]\langle 45\rangle\langle 56\rangle}\right]. \tag{15}$$

The 1236 and 1237 boxes each receive a contribution from a second helicity assignment which we have not drawn here and which is not needed in (13). The other factors appearing in (13) are the usual conversion factors between integral coefficients and box coefficients.

It is not difficult to verify that X_2 satisfies the relation

$$X_2 + F[G[X_2]] = F[X_2] + G[X_2], \tag{16}$$

which renders two of the terms in (10) redundant, as anticipated in (1) (where we use $\overline{\mathrm{flip}}$ to denote the composition $F[G[\,]]$). In conclusion, let us note that we have verified numerically that the formula (1) agrees with the amplitude obtained by summing up the necessary 44 CSW diagrams. As further evidence, it is rather straightforward to check that (1) has all of the correct collinear limits. In particular, it reproduces the simple three-term representation of the 7-particle tree amplitude found in [9], as one might expect from taking the collinear limits of the full box coefficients entering into the calculation.

We would like to thank Zvi Bern for many helpful discussion and correspondence, and especially for sending us the list of IR equations for $n = 8$. R. Roiban is grateful to the KITP for hospitality and support during the QCD and String Theory Workshop, where this work was initiated. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949 (M. S., A. V.), and by the DOE under Grant No. DE-FG02-91ER40671 (R. R.).

[1] E. Witten, Commun. Math. Phys. **252**, 189 (2004).
[2] R. Roiban, M. Spradlin, and A. Volovich, J. High Energy Phys. 04 (2004) 012.
[3] R. Roiban and A. Volovich, Phys. Rev. Lett. **93**, 131602 (2004).
[4] R. Roiban, M. Spradlin, and A. Volovich, Phys. Rev. D **70**, 026009 (2004).
[5] F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 09 (2004) 006
[6] S. Gukov, L. Motl, and A. Neitzke, hep-th/0404085.
[7] I. Bena, Z. Bern, and D. A. Kosower, Phys. Rev. D **71**, 045008 (2005).
[8] Our conventions are mostly standard, except that we use the "twistor" convention for the negative helicity inner product, which differs from the convention of [9] by a sign. In particular, we use $\langle pq\rangle[pq] = 2p\cdot q$, $[p|q|r\rangle = [pq]\langle qr\rangle$, $[p|(q+r)|s\rangle = [p|q|s\rangle + [p|r|s\rangle$, etc., and define $k_i^{[r]} = p_i + \cdots + p_{i+r-1}$ and $t_i^{[r]} = (k_i^{[r]})^2$.
[9] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, Phys. Rev. D **71**, 045006 (2005).
[10] Some savings could be achieved by writing down the amplitude from an "intermediate prescription," as outlined in [7]. Given the miraculous cancellations we observe in this Letter, it is natural to wonder whether there is a similar cancellation between the terms appearing in such an expression, or between some of the 44 CSW diagrams. However, all of these terms individually depend on an arbitrary reference spinor η (which drops out of the full amplitude), and while it is likely that cancellations occur for certain clever choices of η, is it not *a priori* clear how to choose η to maximize the cancellations.
[11] This suggestion has also been made in [12], which appeared as this Letter was in preparation.
[12] Z. Bern, Lance J. Dixon, and D. A. Kosower, hep-th/0412210.
[13] It would be interesting to see how many roots there are to the algebraic equations which de- scribe this amplitude as explained in [4]. Perhaps there is some relation between the number of roots and the number of terms in the most compact expression possible for a given amplitude.
[14] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[15] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[16] A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B706**, 150 (2005).
[17] F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 10 (2004) 077.
[18] M. Luo and C. Wen, J. High Energy Phys. 11 (2004) 004.
[19] I. Bena, Z. Bern, D. A. Kosower, and R. Roiban, hep-th/0410054.
[20] F. Cachazo, hep-th/0410077.
[21] M. Luo and C. Wen, hep-th/0410118.
[22] R. Britto, F. Cachazo, and B. Feng, Phys. Rev. D **71**, 025012 (2005).
[23] C. Quigley and M. Rozali, J. High Energy Phys. 01 (2005) 053.
[24] J. Bedford, A. Brandhuber, B. Spence, and G. Travaglini, Nucl. Phys. **B706**, 100 (2005).
[25] S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon, and D. C. Dunbar, Phys. Lett. B **606**, 189 (2005).
[26] L. J. Dixon, E. W. N. Glover, and V. V. Khoze, J. High Energy Phys. 12 (2004) 015.
[27] R. Britto, F. Cachazo, and B. Feng, hep-th/0412103.
[28] J. Bedford, A. Brandhuber, B. Spence, and G. Travaglini, hep-th/0412108.
[29] Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl. Phys. **B513**, 3 (1998).
[30] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 01 (2000) 027.
[31] Z. Bern, L. J. Dixon, and D. A. Kosower, J. High Energy Phys. 08 (2004) 012.
[32] W. T. Giele and E. W. N. Glover, Phys. Rev. D **46**, 1980 (1992).
[33] Z. Kunszt, A. Signer, and Z. Trocsanyi, Nucl. Phys. **B420**, 550 (1994).
[34] S. Catani, Phys. Lett. B **427**, 161 (1998).
[35] There is one other linearly independent equation with only $(\mathrm{MHV})^4$ boxes: $c_{1245} - c_{1258} - c_{1456} + c_{1568} = 0$. However, the anomaly in the $t_1^{[4]}$ cut can be used to verify that all four of these coefficients are equal to each other (and nonzero), so this equation is not useful.