Unified Tensile Fracture Criterion

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We find that the classical failure criteria, i.e., maximum normal stress criterion, Tresca criterion, Mohr-Coulomb criterion, and von Mises criterion, cannot satisfactorily explain the tensile fracture behavior of the bulk metallic glass (BMG) materials. For a better description, we propose an ellipse criterion as a new failure criterion to unify the four classical criteria above and apply it to exemplarily describe the tensile fracture behavior of BMGs as well as a variety of other materials. It is suggested that each of the classical failure criteria can be unified by the present ellipse criterion depending on the difference of the ratio $\alpha = \tau_0/\sigma_0$.

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The deformation and fracture behavior of various structural materials or matter, such as metals, ceramics, intermetallics, rock, soil, concrete, etc., has been investigated for more than 200 years, and many important theories or rules were proposed and well developed. For brittle materials, for example, ceramics, intermetallics, rock, soil, and concrete, their yield and failure often occur simultaneously due to lack of plasticity or working hardening ability. Among the failure criteria available, there are four classical ones, which have been widely used for the yield or failure of brittle materials, as summarized in some textbooks [1–4].

As illustrated in Figs. 1(a) and 1(b), when one applies a tensile stress σ_T to a specimen, the normal stress σ and the shear stress τ on any shear plane θ can be expressed as

$$\sigma = \sigma_T \sin^2\theta \tag{1a}$$

$$\tau = \sigma_T \sin\theta \cdot \cos\theta. \tag{1b}$$

Obviously, the maximum normal stress $\sigma_{\max}$ occurs on the 90° plane of the specimen. Therefore, the tensile failure condition of the maximum normal stress criterion [1–4] is

$$\sigma_{\max} \geq \sigma_0 \tag{2}$$

at $\sigma_0 < \tau_0$. σ_0 and τ_0 are, respectively, the critical normal fracture stress and the shear fracture stress of the material. The normal stress σ and the shear stress τ on any plane can be described by a Mohr circle in a σ-τ coordinate, as illustrated in Fig. 1(c). Apparently, the tensile failure condition of a specimen results in $\theta_T = 90°$ at $\sigma_T = \sigma_0$.

Besides, the maximum shear stress $\tau_{\max}$ occurs on a plane inclined by 45° with respect to the tensile stress axis, as shown in Fig. 1(b). Therefore, the tensile failure condition of the Tresca criterion is at the maximum shear stress plane, i.e.,

$$\tau_{\max} \geq \tau_0.$$

The tensile failure of a specimen can be illustrated as in Fig. 1(d). In the case of $\tau_0 < \sigma_0$, the tensile shear fracture will occur at $\theta_T = 45°$ according to the Tresca criterion [1–4].

The Mohr-Coulomb criterion [1–4] was proposed in 1773, and it requires that the shear failure does not depend only on the shear stress τ but also on the normal stress σ, i.e.,

$$\tau + \mu \cdot \sigma \geq \tau_0. \tag{4}$$

Therefore, the tensile shear fracture plane will deviate from the maximum shear stress plane of 45°, i.e., $45° < \theta_T < 90°$, as illustrated in Fig. 1(e).

Another classical criterion is the von Mises criterion, which was proposed in 1913 based on the distortion energy theory [1–4], i.e.,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 6Y^2. \tag{5a}$$

Here, σ_1, σ_2, and σ_3 are three principal stresses of the specimen, and Y is a material constant. In the case of a two-dimensional stress state, i.e., $\sigma_3 = 0$, the von Mises criterion can be changed into

$$\sigma^2 + 3\tau^2 \geq 3Y^2. \tag{5b}$$

According to this equation, the tensile failure of a specimen can be schematically illustrated as in Fig. 1(f), where $\sigma_0 = \sqrt{3}\tau_0 = Y$. Therefore, the tensile shear fracture should always result in $\theta_T = 60°$ according to the von Mises criterion [1–4], as shown in the figure.

Altogether, the four classical criteria described above can explain only different tensile failure modes. This raises two crucial questions: (i) which failure criterion is more suitable for various brittle materials? and (ii) is there a unified tensile fracture criterion to describe all the possible failure modes?

For ductile crystalline materials, slip deformation can proceed only on some special low index lattice planes, such

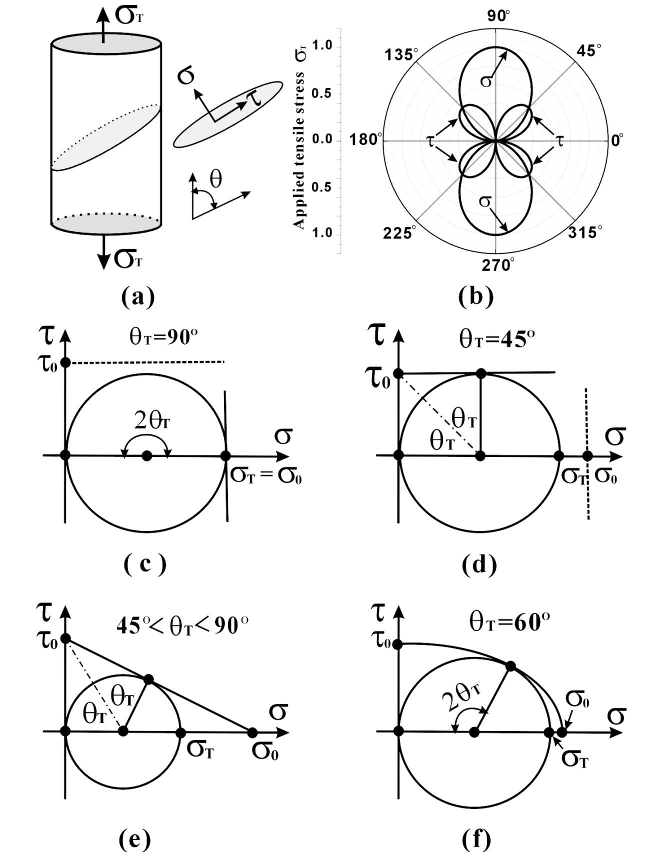

FIG. 1. (a) Illustration of a tensile specimen and shear plane and (b) distribution of normal stress σ and shear stress τ on any shear plane of a tensile specimen. Schematic illustration of tensile failure based on (c) maximum normal stress criterion at $\sigma_0 < 2\tau_0$ and $\theta_T = 90°$, (d) Tresca criterion at $\sigma_0 > 2\tau_0$ and $\theta_T = 45°$, (e) Mohr-Coulomb criterion at $45° < \theta_T < 90°$, and (f) von Mises criterion at $\theta_T = 60°$.

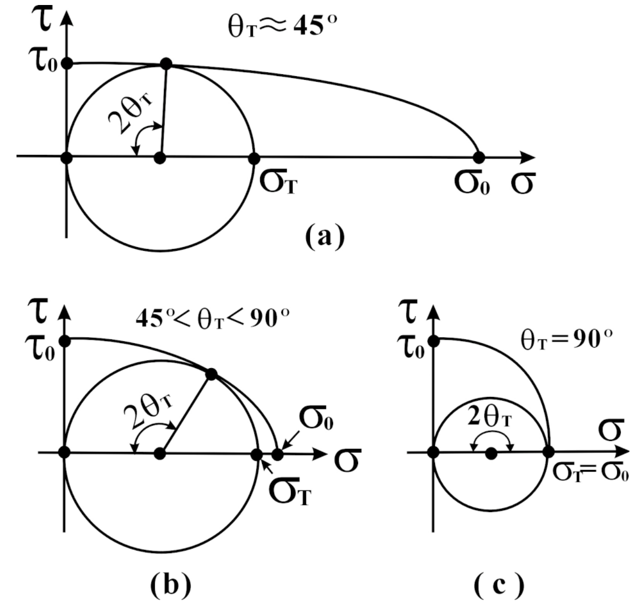

FIG. 2. Schematic illustration of the ellipse criterion at different conditions: (a) $\alpha = \tau_0/\sigma_o \to 0$ or $\sigma_o \to \infty$ and $\theta_T \approx 45°$; (b) $0 < \alpha = \tau_0/\sigma_o < \sqrt{2}/2$ and $45° < \theta_T < 90°$; (c) $\alpha = \tau_0/\sigma_o \geq \sqrt{2}/2$ and $\theta_T = 90°$.

as {111} planes in fcc metals or {0001} planes in hcp metals [3–5]. During the last ten years, a new class of material, i.e., bulk metallic glass (BMG), was discovered and attracted worldwide attention due to its unique properties [6,7]. Their plastic deformation and fracture are always localized in some shear bands with little overall plasticity [8–11]. However, up to now it is not clear for this type of material which is the preferential shear plane (or yield plane) under different loading modes. For example, when a $Zr_{52.5}Ni_{14.6}Al_{10}Cu_{17.9}Ti_5$ BMG is subjected to a tensile load, the shear fracture occurs on a plane of around 56° with respect to the tensile axis rather than 45°, as shown in Fig. 1(c) in the literature [12]. The fracture surface often exhibits radiating veinlike patterns, as shown in Fig. 2(b) in the literature [12]. We have previously proposed that the radiating veinlike patterns give evidence for a promotion effect of the normal stress σ on the tensile shear fracture [12–14]. Besides, many other investigators also observed an obvious deviation of the tensile shear fracture angle from 45°; Table I lists the θ_T values of various BMGs reported so far [8–24], covering several different alloy systems, such as Zr-, Cu-, Fe-, Co-, La-, Al-, Ni-, Pd-based alloys, etc.

The findings outlined above strongly indicate that the deviation of the tensile shear fracture plane from 45° is a common phenomenon in BMGs. However, there is no definite shear fracture plane in these materials, as for example, {111} slip planes in crystals. First, in various BMGs, θ_T obeys the following relationship [12]: i.e., $45° < \theta_T \leq 90°$, indicating that the Tresca criterion is invalid. Second, the maximum normal stress criterion cannot explain why the BMG materials often fail in a shear mode with different fracture angles (see Table I). The von Mises criterion can explain only the case of $\theta_T = 60°$, rather than $45° < \theta_T \leq 90°$, as illustrated in Fig. 1(f). It seems that the Mohr-Coulomb criterion is more suitable to describe the shear failure of BMGs on an arbitrary shear plane [9,12,13,17,19,25]. However, the Mohr-Coulomb criterion cannot explain why a BMG specimen sometimes fails along a plane perpendicular to the stress axis, i.e., $\theta_T = 90°$, as listed in Table I. Therefore, we can conclude that none of the four classical criteria above can satisfactorily explain the tensile fracture modes of BMGs, but it is necessary to develop a new tensile failure criterion, which is more suitable for BMGs or other brittle materials.

Let us come back to the von Mises criterion [Eq. (5b)]. It proposes that the ratio of τ_0/σ_0 is a constant, i.e., $\tau_0/\sigma_0 = \sqrt{3}/3$. However, this constant does not reflect possible differences in the atomic structure or microstructure of different classes of materials. Therefore, if one considers the ratio $\alpha = \tau_0/\sigma_0$ to take an arbitrary value, then

TABLE I. Comparison of the tensile shear fracture angle θ_T for different metallic glasses.

Investigators	Compositions	Fracture angle (θ_T)	Ratio $\alpha = \tau_0/\sigma_0$
Alpas *et al.* [8]	$Ni_{78}Si_{10}B_{12}$	~55°	0.504
Bengus *et al.* [10]	$Fe_{70}Ni_{10}B_{20}$	~60°	0.577
Davis and Yeow [11]	$Ni_{49}Fe_{29}P_{14}B_6Si_2$	~53°	0.464
He *et al.* [14]	$Zr_{52.5}Ni_{14.6}Al_{10}Cu_{17.9}Ti_5$	55°–65°	0.504–0.625
Inoue *et al.* [15]	$Cu_{60}Zr_{30}Ti_{10}$	~54°	0.485
Inoue *et al.* [16]	$(Al_{84}Y_9Ni_5Co_2)_{0.95}Sc_5$	~90°	>0.707
Lee *et al.* [17]	$La_{62}Al_{14}(Cu, Ni)_{24}$	~90°	>0.707
Lewandowski and Lowhaphandu [18]	$Zr_{40}Ti_{12}Ni_{9.4}Cu_{12.2}Be_{26.4}$	~55°	0.504
Liu *et al.* [19]	$Zr_{52.5}Ni_{14.6}Al_{10}Cu_{17.9}Ti_5$	53°–60°	0.464–0.577
Megusar *et al.* [20]	$Pd_{80}Si_{20}$	~50°	0.384
Mukai *et al.* [21]	$Pd_{40}Ni_{40}P_{20}$	~56°	0.522
Noskova *et al.* [22]	$Co_{70}Si_{15}B_{10}Fe_5$	~60°	0.577
Saida *et al.* [23]	$Zr_{80}Pt_{20}$	~90°	>0.707
Takayama [24]	$Pd_{77.5}Cu_6Si_{16.5}$	~51°	0.414
Zhang *et al.* [12]	$Zr_{52.5}Ni_{14.6}Al_{10}Cu_{17.9}Ti_5$	~56°	0.522
Zhang *et al.* [13]	$Zr_{59}Cu_{20}Al_{10}Ni_8Ti_3$	~54°	0.485

Eq. (5b) can be expressed in a unified form, i.e.,

$$\frac{\sigma^2}{\sigma_0^2} + \frac{\tau^2}{\tau_0^2} \geq 1. \quad (6)$$

We define Eq. (6) as a new tensile failure criterion, hereafter named "ellipse criterion." This new criterion indicates that tensile failure is controlled by both the normal stress σ and the shear stress τ. But the dependence of shear stress τ on normal stress σ [Eq. (6)] is not linear, as the Mohr-Coulomb criterion [9,12,13,17,19,25]. An illustration of the ellipse criterion is shown in Figs. 2(a)–2(c). The shape of the ellipse depends on the ratio $\alpha = \tau_0/\sigma_0$ and can be classified into four cases. When $\alpha = \tau_0/\sigma_0 \to 0$ or $\sigma_0 \to \infty$, θ_T should be quite close to 45°, which is consistent with the Tresca criterion [see Fig. 1(d)]. When $0 < \alpha = \tau_0/\sigma_0 < \sqrt{2}/2$, θ_T will range from 45° to 90°, which agrees well with the Mohr-Coulomb criterion. For $\alpha = \tau_0/\sigma_0 = \sqrt{3}/3$, one gets $\theta_T = 60°$, indicating that the von Mises criterion is a special case of the present ellipse criterion. Finally, when $\alpha = \tau_0/\sigma_0 \geq \sqrt{2}/2$, the tensile fracture will always occur along the plane perpendicular to the tensile axis, i.e., $\theta_T = 90°$ [see Fig. 2(c)]. This means that the maximum normal stress criterion is also one of the special cases for the present ellipse criterion.

Figure 3 shows the dependence of σ_T and θ_T on the ratio $\alpha = \tau_0/\sigma_0$. Two points A and B are marked on the σ_T and θ_T lines of the figure. Point A represents the position of the Tresca criterion at $\alpha = \tau_0/\sigma_0 \to 0$. This results in $\theta_T \to 45°$ and $\sigma_T = 2\tau_0$. Point B represents the position of the von Mises criterion at $\alpha = \tau_0/\sigma_0 = \sqrt{3}/3$, in this case, $\theta_T = 60°$ and $\sigma_T = 2\sqrt{2/3}\tau_0$. In addition, there are two regions marked by C and D in the figure. Region C contains the Mohr-Coulomb fracture range at $0 < \alpha = \tau_0/\sigma_0 < \sqrt{2}/2$ and $45° < \theta_T < 90°$. Region D corresponds to the range of the maximum normal stress criterion at $\alpha = \tau_0/\sigma_0 \geq \sqrt{2}/2$ and $\theta_T = 90°$. Based on the analysis above, σ_T and θ_T can be expressed as functions of $\alpha = \tau_0/\sigma_0$. When $0 < \alpha = \tau_0/\sigma_0 < \sqrt{2}/2$,

$$\sigma_T = 2\tau_0\sqrt{1 - \alpha^2}, \quad (7a)$$

$$\theta_T = \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{\sqrt{1 - 2\alpha^2}}{\alpha^2}\right). \quad (7b)$$

When $\alpha = \tau_0/\sigma_0 \geq \sqrt{2}/2$,

$$\sigma_T = \sigma_0 = \tau_0/\alpha, \quad (8a)$$

$$\theta_T = 90°. \quad (8b)$$

Since θ_T strongly depends on the ratio $\alpha = \tau_0/\sigma_0$, we can

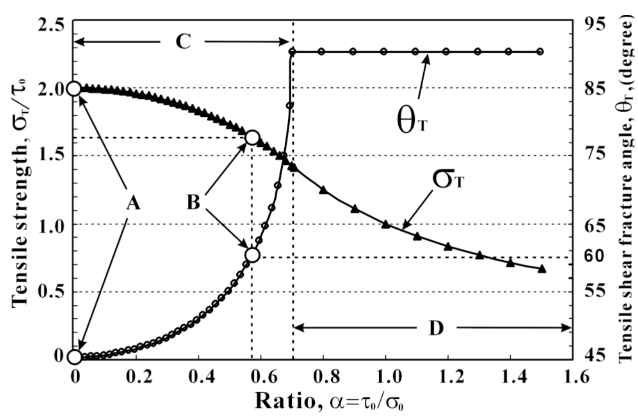

FIG. 3. Dependence of tensile fracture strength σ_T and tensile shear fracture angle θ_T on the ratio $\alpha = \tau_0/\sigma_o$ according to the ellipse criterion. In a different range of $\alpha = \tau_0/\sigma_o$, each of the four classical failure criteria can be unified by the present ellipse criterion.

define $\alpha = \tau_0/\sigma_0$ as the factor determining the fracture mode. Therefore, the details of the failure modes of a material are controlled by the factor $\alpha = \tau_0/\sigma_0$ according to the unified criterion.

In summary, the four well-established classical failure criteria can be unified by the present ellipse criterion in terms of the variation of $\alpha = \tau_0/\sigma_0$. The ratio $\alpha = \tau_0/\sigma_0$ reflects the difference in the bonding property between atoms of different materials and, in return, controls the macroscale fracture modes of the materials. When $\alpha = \tau_0/\sigma_0 \rightarrow 0$, or τ_0 is extremely low, the ellipse criterion is equivalent to the Tresca criterion and might be also suitable for the slip deformation in ductile crystalline materials, which obeys the Schmid law [3–5]. For example, the critical resolved shear stress is nearly independent of the orientation for single crystals [5]. When $0 < \alpha = \tau_0/\sigma_0 < \sqrt{2}/2$, θ_T is in the range of 45°–90° and is consistent with Mohr-Coulomb and von Mises criteria. From the data in Table I, where θ_T mainly varies between 50° and 65° for various BMGs, we can derive $\alpha = \tau_0/\sigma_0 = 0.384$–$0.625$. This indicates that the extremely high strength (1–5 GPa) of various BMGs [8–24] can be attributed to the great increase in the shear resistance τ_0 in comparison with the critical resolved shear stress of the crystalline materials. When $\alpha = \tau_0/\sigma_0 \geq \sqrt{2}/2$, the maximum normal stress $\sigma_{\max}$ will control the failure mode, resulting in $\theta_T = 90°$, which is consistent with the maximum normal stress criterion. This might explain why there is often a big asymmetry between tensile and compressive strength and why cleavage failure is easier than shear fracture in some brittle materials, such as rock, intermetallics, ceramics, etc. [1–4]. Therefore, in terms of the variation of $\alpha = \tau_0/\sigma_0$ (see Figs. 2 and 3), most materials can be divided into three types: (i) ductile crystalline materials with low strength at $\alpha = \tau_0/\sigma_0 \rightarrow 0$, (ii) high-strength materials, such as BMGs or nanostructured materials at $\alpha = \tau_0/\sigma_0 \approx 1/3$–$2/3$, and (iii) brittle materials with high hardness at $\alpha = \tau_0/\sigma_0 \geq \sqrt{2}/2$, such as rock, intermetallics, ceramics, etc. Furthermore, due to the great difference in the ratio $\alpha = \tau_0/\sigma_0$, the materials in the world can display quite different fracture modes and strength.

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