Physical Bounds to the Entropy-Depolarization Relation in Random Light Scattering

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We present a theoretical study of multimode scattering of light by optically random media, using the Mueller-Stokes formalism which permits us to encode all the polarization properties of the scattering medium in a real 4×4 matrix. From this matrix two relevant parameters can be extracted: the depolarizing power D_M and the polarization entropy E_M of the scattering medium. By studying the relation between E_M and D_M, we find that *all* scattering media must satisfy some *universal* constraints. These constraints apply to both classical and quantum scattering processes. The results obtained here may be especially relevant for quantum communication applications, where depolarization is synonymous with decoherence.

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Introduction.—Optical properties of random media have drawn quite a bit of interest in recent years: since a light field is a vector wave, this includes coverage of the polarization aspects [1]. When polarized light is incident on an optically random medium it suffers multiple scattering and, as a result, it may emerge partly or completely depolarized. The amount of depolarization can be quantified by calculating either the entropy (E_F) or the degree of polarization (P_F) of the scattered field [2]. It is simple to show that the field quantities E_F and P_F are related by a single-valued function: $E_F = E_F(P_F)$. For example, polarized light ($P_F = 1$) has $E_F = 0$ while partially polarized light ($0 \leq P_F < 1$) has $1 \geq E_F > 0$. When the incident beam is polarized and the output beam is partially polarized, the medium is said to be depolarizing. An average measure of the depolarizing power of the medium is given by the so-called depolarization index (D_M) [3]. Nondepolarizing media are characterized by $D_M = 1$, while depolarizing media have $0 \leq D_M < 1$. A depolarizing scattering process is always accompanied by an increase of the entropy of the light, the increase being due to the interaction of the field with the medium. An average measure of the entropy that a given random medium can add to the entropy of the incident light beam is given by the polarization entropy E_M [4]. Nondepolarizing media are characterized by $E_M = 0$, while for depolarizing media $0 < E_M \leq 1$. As the field quantities E_F and P_F are related to each other, so are the medium quantities E_M and D_M with the key difference that, as we show later, E_M is a *multivalued* function of D_M.

The purpose of this Letter is to point out a universal relation between the polarization entropy E_M and the depolarization index D_M valid for any random scattering medium. This relation covers the complete regime from zero to total depolarization. It has been introduced before, by Le Roy–Brehonnet and Le Jeune [4], in an empirical sense, to classify depolarization measurements on rough surfaces (sand, rusty steel, polished steel, etc.). We derive here its theoretical foundation and present analytical expressions for the multivalued function $E_M = E_M(D_M)$. Although the (E_M, D_M) relation is essentially classical, we use a single-photon theoretical approach, exploiting the well known analogy between single-photon and classical optics [5]. We prefer this to a classical formulation since it offers a natural starting point for the extension to entangled twin-photon light scattering by a random medium, which *is* a true quantum phenomenon that could deteriorate quantum communication. Moreover, the results obtained here, although derived within the context of quantum and classical optics, could have been equally well developed in other contexts as, e.g., particle physics or statistical mechanics [6], since the presence of two-level systems (as is the polarization of a photon) and decoherence processes (as depolarization) is almost ubiquitous in physics.

Polarization description of the field.—Let us consider a collimated light beam propagating in the direction z. In a given spatial point $\mathbf{r}$, the quasimonochromatic time-dependent electric field associated with the beam is a complex-valued vector $\mathbf{E}(t) = X(t)\boldsymbol{x} + Y(t)\boldsymbol{y}$. This vector defines the instantaneous polarization of the light which is, in any short enough time interval, fully polarized. Alternatively, the same light beam may be described by a time-dependent real-valued unit Stokes vector $\boldsymbol{s}(t) = \{2\,\mathrm{Re}(X^*Y), 2\,\mathrm{Im}(X^*Y), |X|^2 - |Y|^2\}/(|X|^2 + |Y|^2)$, which moves on the Poincaré sphere (PS) [7]. Of course, no detector can measure the instantaneous polarization; the best one can get is an average polarization over some time interval T. If during the measurement time T the Stokes vector $\boldsymbol{s}(t)$ maintains the same direction, then the beam is polarized. Vice versa, if $\boldsymbol{s}(t)$ moves over the PS covering some finite area, then the beam is partially polarized. In the last case, for stationary beams, the motion of $\boldsymbol{s}(t)$ produces a probability distribution over the PS which determines the degree of polarization of the light [8]. Time dependence of the polarization is not the only cause for depolarization; also spatial dependence, for example, may lead to loss of polarization.

We stress that this picture is not limited to the classical domain; in Ref. [9] we found, e.g., that a multimode single-photon scattering process generates a $\mathbf{k}$-dependent Stokes vector distribution. More generally, if $\psi = \{t, \mathbf{k}, \lambda, \ldots\}$ denotes the set of all variables (e.g., time t, momentum $\mathbf{k}$, polarization λ, etc.) on which $\mathbf{s} = \mathbf{s}(\psi)$ depends, then the state of a polarized light beam (either classical or quantum) may be described by a 2×2 matrix $\rho(\psi) = [\sigma_0 + \mathbf{s}(\psi) \cdot \boldsymbol{\sigma}]/2$, where σ_0 is the 2×2 identity matrix and $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are the Pauli matrices. The matrix $\rho(\psi)$ is known as the *coherency* matrix in classical optics [7] and as the *density* matrix in quantum mechanics [10]. Since by construction $\mathrm{Tr}\rho(\psi) = 1$, each matrix $\rho(\psi)$ can describe either a *purely* polarized beam in classical optics or a *pure* photon state in quantum optics [11]. However, the state of a partially polarized beam must be described by the matrix $\rho = \int (d\psi) w(\psi) \rho(\psi)$, where $\int (d\psi)$ is the integration measure [12] in the space of the variables ψ and $\int (d\psi) w(\psi) = 1$. The statistical weight $w(\psi) \geq 0$ defines a probability distribution over the PS. It is clear that ρ can represent a *mixed* photon state in the context of quantum optics as well. If A denotes any polarization-dependent observable, its average value must be calculated as $\langle A \rangle = \mathrm{Tr}(\rho A) = \int (d\psi) w(\psi) \mathrm{Tr}(A \rho(\psi))$. If A represents the entropy of the field, i.e., $A = -\ln(\rho)$, then $\langle A \rangle = -\mathrm{Tr}(\rho \ln \rho)$, which is the von Neumann entropy S of the photon state [13]. However, it is easy to see that this coincides with the Gibbs entropy [14] of the distribution $w(\psi)$, since $S = -\int (d\psi) w(\psi) \ln(w(\psi))$, in agreement with the results of Ref. [8].

Single-photon scattering and multimode Mueller formalism.—The theoretical framework for studying one-photon scattering has been established elsewhere [9], and here we use the results found in [9] to extend the Mueller-Stokes formalism to quantum scattering processes. In classical optics a polarization-dependent scattering process can be characterized by a real-valued 4×4 matrix, the so-called Mueller matrix M [2], which describes the polarization properties of the scattering medium. We show now that such a matrix description can be extended to the quantum (single-photon) scattering case. Let us consider a photon prepared in the pure state $\rho(\psi)$, approximately described by a monochromatic plane wave $|\mathbf{k}_0, \lambda_0\rangle$. In this case $\psi = \{\mathbf{k}_0, \lambda_0\}$. Now, let us suppose that the photon is transmitted through a linear optical system described by a unitary scattering operator T such that $\rho(\psi') = T \rho(\psi) T^\dagger$ represents the pure state of the photon after the scattering, where ψ' is the set of *all* scattered modes: $\psi' = \{\mathbf{k}_1, \lambda_1, \mathbf{k}_2, \lambda_2, \ldots\}$. A multimode detection scheme implies a reduction from the set ψ' to the subset of the *detected* modes $\psi'' = \{\mathbf{k}_1, \lambda_1, \ldots, \mathbf{k}_N, \lambda_N\} \subset \psi'$ which causes a transition from the pure state $\rho(\psi')$ to the mixed state $\rho = \int (d\psi'') w(\psi'') \rho(\psi'')$. If we denote the Stokes parameters of the beam before and after the scattering with $s_\mu = \mathrm{Tr}(\rho(\psi)\sigma_\mu)$ and s'_μ, respectively ($\mu = 0, 1, 2, 3$), then the classical result $s'_\mu = \sum_{\nu=0}^{3} M_{\mu\nu} s_\nu$ is retrieved, with the difference that a *generalized* (measured) Mueller matrix $\|M_{\mu\nu}\|$ appears, which is defined as

$$M_{\mu\nu} \propto \int_{\psi''} d\mathbf{k}\, m_{\mu\nu}(\mathbf{k}). \tag{1}$$

The local (with respect to the momentum) matrix elements $m_{\mu\nu}(\mathbf{k})$ are defined by means of the matrix relation

$$W^T(\mathbf{k}) T^\dagger(\mathbf{k}, \mathbf{k}_0) \sigma_\mu T(\mathbf{k}_0, \mathbf{k}) W(\mathbf{k}) = m_{\mu\nu}(\mathbf{k}) \sigma_\nu \tag{2}$$

($\mu, \nu = 0, 1, 2, 3$), and summation over repeated indices is understood. Explicit expressions for the 2×2 matrices $W(\mathbf{k})$ and $T(\mathbf{k}, \mathbf{k}')$ can be found in Ref. [9]. The proportionality factor in Eq. (1) can be fixed by imposing the condition $M_{00} = 1/2$. When ψ'' reduces to a single mode $\{\mathbf{k}, \lambda\}$, then $W_{ij}(\mathbf{k}) = \delta_{ij}$ and the classical formalism is fully recovered.

Depolarization index D_M and polarization entropy E_M.—Now that we have a recipe to calculate the Mueller matrix describing a multimode scattering process, we use this knowledge to study the depolarization properties of the scattering medium. Within the Mueller-Stokes formalism, the degree of polarization P_F of the field and the depolarization index D_M of the medium are defined as $P_F = (s_1^2 + s_2^2 + s_3^2)^{1/2}/s_0$ and $D_M = [\mathrm{Tr}(M^T M)/3 - 1/3]^{1/2}$, respectively, where s_μ ($\mu = 0, 1, 2, 3$) are the Stokes parameters of the field and $M_{00} = 1/2$ has been assumed. A deeper characterization of the scattering medium can be achieved by using the Hermitian matrix H [15,16] defined as $H = \sum_{\mu,\nu}^{0,3} M_{\mu\nu}(\sigma_\mu \otimes \sigma_\nu^*)/2$, where $\mathrm{Tr}(H) = 1$. The matrix H has a straightforward physical meaning: $H_{\alpha\beta} = \langle T_{ij} T_{kl}^* \rangle$, where $\alpha = 2i + j$, $\beta = 2k + l$ ($i, j, k, l = 0, 1$). The T_{ij} are the elements of the scattering (Jones) matrix T and brackets indicate the average over the statistical ensemble describing the medium [17]. Then it is clear [4] that a physically realizable optical system is characterized by a positive-semidefinite matrix H. Let $0 \leq \lambda_\nu \leq 1$ ($\nu = 0, \ldots, 3$) be the eigenvalues of H; it can be shown that both the depolarization index D_M and the polarization entropy E_M are simple functions of the λ_ν's. Explicitly we have

$$D_M = \left[\left(4 \sum_{\nu=0}^{3} \lambda_\nu^2 - 1 \right) / 3 \right]^{1/2}, \tag{3}$$

$$E_M = -\sum_{\nu=0}^{3} \lambda_\nu \log_4(\lambda_\nu). \tag{4}$$

Now we are ready to show the universal character of the (E_M, D_M) plot originally introduced in Ref. [4]. More precisely, we show that it allows one to characterize *all* possible scattering media by means of their polarimetric properties. The main idea is the following: both E_M and D_M depend on the four real eigenvalues of H which actually reduces to three independent variables because of the trace constraint $\mathrm{Tr}(H) = 1$. If we eliminate one of these variables in favor of D_M we can write $E_M = E_M(D_M, \alpha, \beta)$ where α, β represent the last two independent variables. Then, for each value of $0 \leq D_M \leq 1$, different values of E_M can be obtained by varying α and β between 0 and 1. In such a way we obtain a whole domain in the D_M-E_M plane instead of just a curve. In order to do that, we have implemented a Monte Carlo code to generate a uniform distribution of points over the four-dimensional unit sphere: the square of the four coordinates of each point is an admissible set of eigenvalues of H. In this way we have generated the graph shown in Fig. 1. The boundary of this domain is formed by the curves C_{ij} $(i, j = 1, \ldots, 4)$, joining the points $(p_i \rightarrow p_j)$. The analytical expressions for these curves are

$$E(n, f) = -[(1 - nf)\log_4(1 - nf) + nf\log_4(f)], \quad (5)$$

where $f_\pm = [1 \pm \sqrt{1 - 3(n+1)(1 - D_M^2)/(4n)}]/(n+1)$, and $n = 1, 2, 3$ is the number of equal eigenvalues of H (order of degeneracy). The links between the functions $E(n, f)$ and the curves C_{ij} are given in Table I where we have defined $E_{13} = -(1-\mu)\log_4(\frac{1-\mu}{2}) - \mu\log_4(\frac{\mu}{2})$. The curve C_{14} is special in the sense that it sets an upper bound for the entropy of *any* scattering medium. We find numerically that the value of the entropy on this curve is very well approximated by

$$E_M^{\rm cr} \sim (1 - D_M^2)^\gamma, \quad (6)$$

where $\gamma \cong 0.862$, which is, interestingly, almost equal to e/π. Then, for all depolarizing scattering media the condition $E_M \lesssim E_M^{\rm cr}$ must be satisfied. It is interesting to note that a purely depolarizing scattering medium (with diagonal Mueller matrix) leads to $E_M \cong E_M^{\rm cr}$. By using thermodynamics language, one may interpret Fig. 1 as a polarization "state diagram" where different phases of a generic scattering medium, characterized by different symmetries of the corresponding Mueller matrices, are separated by the curves C_{ij}. It is worth noting again that there is nothing inherently quantum in the above derivation of the physical bounds Eq. (5), therefore these results have validity both in the classical and in the quantum regimes.

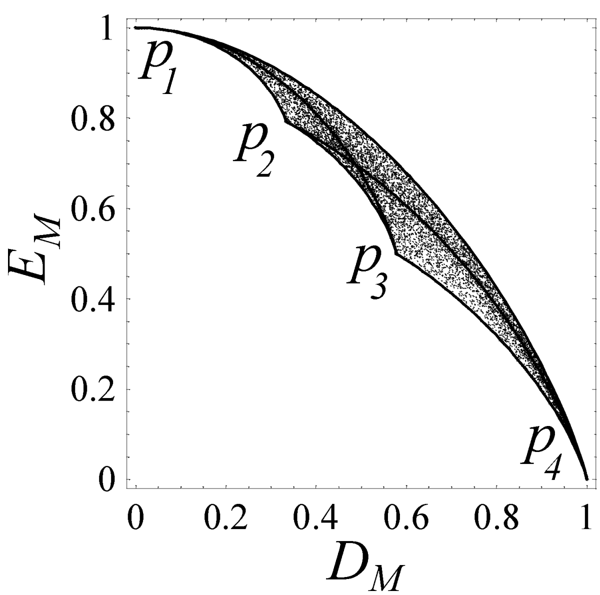

FIG. 1. Numerically determined domain in the D_M-E_M plane corresponding to all physically realizable polarization scattering processes. The solid curves are the analytically obtained bounds. The four cusp points $p_1 = (0, 1)$, $p_2 = (1/3, \log_4 3)$, $p_3 = (1/\sqrt{3}, 1/2)$, $p_4 = (1, 0)$ separate different polarization scattering processes, as described in the text.

Random-matrix approach.—We have checked the validity of the theory outlined above, for scattering media in the regime of applicability of the random-matrix theory (RMT) [18]. Random media, either disordered media [1] or chaotic optical cavities [19], can be represented by ensembles [17]. The transmission of polarized light through a random medium may decrease the degree of polarization in a way that depends on the number N of the detected modes via Eq. (1). Under certain conditions, RMT can account for a statistical description of the light scattering by random media [20,21]. Let $\psi_\lambda(\mathbf{k})$ be the complex probability amplitude that a photon is scattered in the state $|\mathbf{k}, \lambda\rangle$. Then, according to RMT, the real and the imaginary parts of the scattering amplitudes $\psi_\lambda(\mathbf{k})$ are independent Gaussian random variables with zero mean and variance that can be fixed to 1. The assumption of independent variables is justified since usually the set ψ'' of the detected modes is much smaller than the set ψ' of *all* the scattered modes [22]. Let us suppose now that the impinging photon is in the pure state $|\mathbf{k}_0, \lambda_0\rangle$. In this case $\psi_\lambda(\mathbf{k}) = T_{\lambda\lambda_0}(\mathbf{k}, \mathbf{k}_0)$ and the statistical distribution of the $M_{\mu\nu}$'s can be numerically calculated according to Eqs. (1) and (2). In this way we have calculated the

TABLE I. List of the analytical curves (continuous lines) in Fig. 1. The second column refers to the equations generating the corresponding curves, while the third column gives the eigenvalues of H. The first four curves form the boundary of the physical domain; the last two represent inside curves. To each function $E(n, f)$ corresponds a sequence of eigenvalues of H; e.g., $E(2, f_+) \leftrightarrow \{\lambda, \mu, \mu, 0\}$. For each sequence we use the constraint $\mathrm{Tr}(H) = 1$ to write $\lambda + n\mu = 1 \Rightarrow \lambda = 1 - n\mu$ so that μ is the only independent variable left. Then a given sequence can be put in Eq. (3), which can be inverted in order to obtain a function $\mu = \mu(D_M)$. In general, this inversion cannot be done on the whole range $[0, 1]$ of D_M, but only within the subinterval $[p_i, p_j]$ delimited by the cuspidal points $\{p_i\}$. These points are therefore obtained by studying the domain of existence of $\mu(D_M)$. Finally, we put the considered eigenvalues sequence [with $\mu \rightarrow \mu(D_M)$] in Eq. (4), obtaining $E(n, f)$. Note that since $f_+ + f_- = 2/(n+1)$, if $n = 1$ from Eq. (5) follows that $E(1, f_+) = E(1, f_-)$.

Curve	Generating equation	Eigenvalues of H
C_{12}	$E(3, f_+)$	$\{\lambda, \mu, \mu, \mu\}$
C_{23}	$E(2, f_+)$	$\{\lambda, \mu, \mu, 0\}$
C_{34}	$E(1, f_\pm)$	$\{\lambda, \mu, 0, 0\}$
C_{14}	$E(3, f_-)$	$\{\lambda, \mu, \mu, \mu\}$
C_{13}	E_{13}	$\{\lambda, \lambda, \mu, \mu\}$
C_{24}	$E(2, f_-)$	$\{\lambda, \mu, \mu, 0\}$

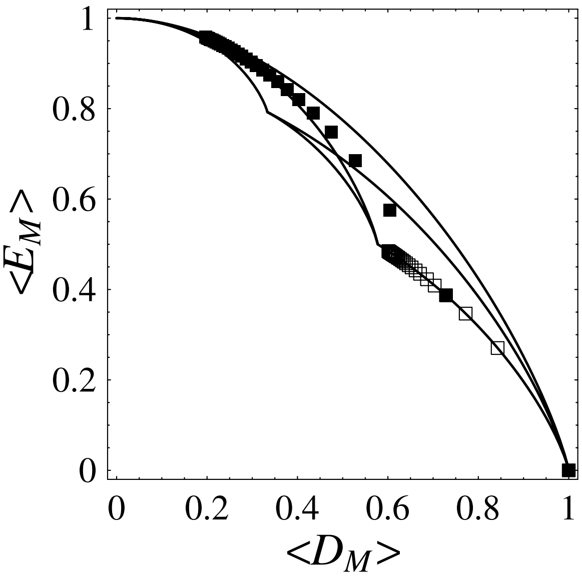

FIG. 2. RMT results for the ensemble-averaged polarization entropy $\langle E_M \rangle$ as a function of the ensemble-averaged depolarization index $\langle D_M \rangle$ for generic (dark squares) and polarization-conserving (open squares) scattering processes. In both cases each point correspond to a given number N of detected modes. When N increases from 1 to 30, points move from the bottom to the top of the figure. The solid lines are the analytical bounds of Fig. 1.

ensemble-averaged polarization entropy $\langle E_M \rangle$ and depolarization index $\langle D_M \rangle$ of the medium, as functions of N for the case in which the angular aperture of the detector is so small that $W_{ij}(\mathbf{k}) \simeq \delta_{ij}$. The results are shown in Fig. 2 for the cases of a generic scattering medium [$T_{\lambda\lambda_0}(\mathbf{k}, \mathbf{k}_0)$ unconstrained] and of a polarization-conserving medium [$T_{\lambda\lambda_0}(\mathbf{k}, \mathbf{k}_0) \propto \delta_{\lambda\lambda_0}$]. The last case is realized when the geometry of the scattering process is confined in a plane. As one can see, for both cases RMT results cover only a small part of the (E_M, D_M) diagram; however, the numerical data are consistent with the analytical bounds given by Eq. (5).

Conclusions.—In summary, we have studied the scattering of light by optically random media, from a polarization point of view. After the calculation of the Mueller matrix M characterizing the polarization properties of a generic scattering medium, we have extracted from M the depolarization index D_M and the polarization entropy E_M. By analyzing the functional relation between E_M and D_M, we have found that the depolarization properties of any scattering medium are constrained by some physical bounds. These bounds have a *universal* character, and they hold in both the classical and the quantum regimes. Our results provide a deeper insight into the nature of random light scattering by giving a useful tool, both to theoreticians and to experimenters, to classify scattering media according to their depolarization properties; we have demonstrated this very recently in a series of experiments on various scattering media [23]. The use of this tool may be particularly relevant in quantum communication where depolarization corresponds to decoherence [24].

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