Generalized Anisotropic Scaling Theory and the Transverse Meissner Transition

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We consider a depinning transition in vortex systems with columnar disorder and tilted applied magnetic fields. From scaling arguments and Monte Carlo simulations, we find that this transverse Meissner transition is governed by a fixed point which is anisotropic in *all* three directions. This generalization of conventional anisotropic scaling means that the correlation length in different directions diverges with different rates, and we derive exact results for the anisotropy exponents. We make predictions which can be tested in experiments on superconductors with columnar disorder.

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Correlated disorder often leads to phase transitions of new universality classes with anisotropic scaling properties. On approaching the phase transition, the correlation length in *one* certain direction might diverge with a different exponent than in the perpendicular directions [1–3]. Here we present a generalization of anisotropic scaling where *all* three directions acquire different scaling dimensions, and apply it to the transverse Meissner (TM) transition in superconductors with columnar defects and a magnetic field applied at a finite angle with respect to the columns.

Phase transitions with anisotropic scaling may occur in systems where one direction is singled out, either by applied fields or by correlated disorder. This is, for example, the case for disordered quantum phase transitions since, in a path integral representation of the partition function, static point-correlated disorder maps to perfectly correlated disorder in the imaginary time direction [3]. Another example is given by the Bose glass transitions in superconductors with columnar defects parallel to a magnetic field, where the correlation length along the columns diverges as $\xi_z \sim \xi^2$ [1,4]. The corresponding quantum phase transition is the well known dirty boson localization transition [2,5].

This Letter studies the TM transition that takes place in superconductors with columnar disorder and a magnetic field applied at a finite tilt angle with respect to the columns. Very little is known about the critical properties at the TM transition in three dimensions, which we demonstrate below to be very different from those of the Bose glass transition that occurs at zero tilt. When the applied magnetic field is slightly tilted away from the columns, the vortex lines stick to the columns until a higher transverse field, i.e., larger tilt angle, is reached at which they depin and become collinear with the magnetic field. In the boson analogy, the transverse field maps to an imaginary vector potential, leading to a localization problem in non-Hermitian quantum mechanics [6].

The TM effect has been studied experimentally [7–14]. The phase diagram for small tilt angles has been determined and shows a characteristic cusp around zero tilt [4,15,16]. However, analysis of experimental results for the critical behavior has been seriously hampered by the lack of a correct scaling ansatz for the transition. In this Letter we derive scaling properties for the TM transition and obtain a correlation length behaving differently in all *three* directions, resulting in a correlation volume with shape $\xi^2 \times \xi \times \xi^3$. This scaling behavior is independently confirmed using Monte Carlo simulations. Our results have striking experimental consequences for the scaling of various physical quantities at the transition that are described below.

Scaling analysis.—Our scaling analysis applies to a class of Bose glass models with correlated disorder and finite range interactions between the vortices. We assume that the columnar defects and a strong magnetic field $H_z = H_\parallel$ are in the z direction, and a weaker transverse field $H_x = H_\perp$ is in the x direction; see Fig. 1.

The presence of a transverse field makes all three directions inequivalent and thus opens the possibility that the correlation length in different directions diverges with different rates as the transition is approached. We therefore make the ansatz $\xi_y = \xi \sim |T - T_c|^{-\nu}$, $\xi_x \sim \xi^\chi$, $\xi_z \sim \xi^\zeta$. The diverging correlations also lead to critical slowing down with a correlation time $\tau \sim \xi^z$. Accordingly, we make an anisotropic scaling ansatz for the singular part

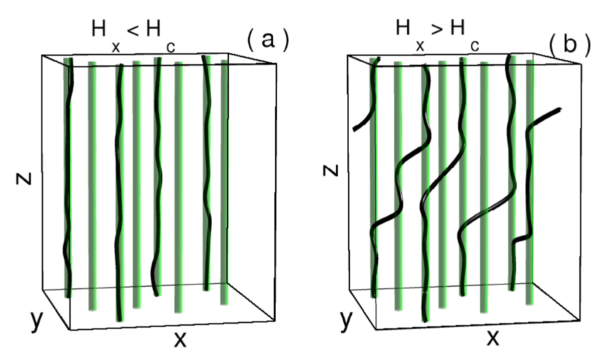

FIG. 1 (color online). (a) Vortex lines localized at the columnar defects in the transverse Meissner phase. (b) The vortex liquid phase, where the vortex lines follow the field direction.

of the disorder-averaged free energy density,

$$f_s(t, h_x, H_y, H_z) \sim l^{-(\chi+1+\zeta)} f_s(l^{1/\nu} t, l^\chi h_x, l H_y, l^\zeta H_z), \quad (1)$$

where l is an arbitrary scale factor, $t = T - T_c$ and $h_x = H_x - H_c^\perp(T)$. The scaling of the magnetic field as the inverse of length, $H_i \sim 1/l_i$, in Eq. (1) follows from gauge invariance, since it enters as an imaginary vector potential for the bosons [15]. Differentiating Eq. (1) immediately leads to scaling laws for the flux density $B_i = -\partial f/\partial H_i$ and the magnetic permeability $\mu_i = \partial B_i/\partial H_i$,

$$B_x \sim l^{-(1+\zeta)}, \qquad \mu_x \sim l^{\chi-1-\zeta}, \quad (2)$$

$$B_y \sim l^{-(\chi+\zeta)}, \qquad \mu_y \sim l^{1-\chi-\zeta}, \quad (3)$$

$$B_z \sim l^{-(\chi+1)}, \qquad \mu_z \sim l^{\zeta-\chi-1}. \quad (4)$$

At the Bose glass transition in a parallel field $\chi = 1$ and $\zeta = 2$, but with a tilted field this is no longer true.

Although the system is not rotationally invariant in the presence of disorder, there is a *statistical* rotation invariance; i.e., a rotation of the transverse field in the xy plane should leave the disorder-averaged free energy unchanged. Imagine therefore doing an infinitesimal rotation $H_x \to H_\perp(1 - \epsilon^2/2)$, $H_y \to H_\perp \epsilon$, leading to

$$\delta f = f\left(h_x - \frac{\epsilon^2 H_\perp}{2}, \epsilon H_\perp\right) - f(h_x, 0) = -\frac{\epsilon^2 H_\perp}{2}\frac{\partial f}{\partial H_x} + \epsilon H_\perp \frac{\partial f}{\partial H_y} + \frac{\epsilon^2 H_\perp^2}{2}\frac{\partial^2 f}{\partial H_y^2} + O(\epsilon^3). \quad (5)$$

Equating this to zero and identifying the coefficients of ϵ and ϵ^2 gives $B_y = 0$ and

$$B_x = \mu_y H_\perp. \quad (6)$$

This identity together with Eqs. (2) and (3) gives $\chi = 2$.

Following Ref. [2] we argue that the magnetic permeability in the z direction μ_z (compressibility in the bosonic language) remains finite at the transition, i.e., $\mu_z \sim \xi^0$. This is the case for an ordinary Bose glass transition [2,5] and our Monte Carlo (MC) data support that it is finite also on the tilted transition. Comparing with Eq. (4) then gives the exponent identity $\zeta - \chi - 1 = 0$. With $\chi = 2$ we then obtain $\zeta = 3$. We call this result 1-2-3 scaling.

We continue by studying the critical exponent ν. The Harris criterion and the Chayes inequality put certain bounds on ν [17]. As a consequence of the highly anisotropic 1-2-3 scaling calculated above, the Harris criterion has to be modified. Assume that there is a local critical temperature $T_c(\mathbf{r})$ in the sample. In order for the transition to be well defined, the variations in $\Delta T_c(\mathbf{r}) = (T_c(\mathbf{r}) - T_c)$ averaged over a correlation volume $\Lambda \sim \xi^{\chi+1+\zeta}$ have to be smaller than $(T - T_c)$ when $T \to T_c$. (T_c is the global critical temperature).

If the disorder is correlated in the z direction but uncorrelated (over large enough distances) in the xy plane the condition becomes

$$\left[\frac{1}{\Lambda^2}\int_\Lambda d^3 r \int_\Lambda d^3 r' \Delta T_c(\mathbf{r}) \Delta T_c(\mathbf{r}')\right]^{1/2} \sim \frac{1}{\xi^{(\chi+1)/2}} \ll |T - T_c| \sim \xi^{-1/\nu}. \quad (7)$$

Close to T_c this is true if $\nu \geq 2/(\chi + 1) = 2/3$.

Model and Monte Carlo simulations.—To calculate ν and to test the scaling predictions we perform MC simulations. We use the Bose glass model with an applied transverse magnetic field. The Bose glass partition function is analogous to a path integral for disordered two-dimensional bosons in imaginary time [1,2,5,18]. We use the superconductor language and call the line degrees of freedom a vorticity. This model is relevant for disordered high temperature superconductors far from the upper critical field H_{c2} [18,19]. The partition function and the Hamiltonian of the short range interaction, lattice version of the model described in Refs. [1,2,18] is $Z = \mathrm{Tr}\exp(-H/T)$ with Hamiltonian

$$H = J \sum_{\mathbf{r}, i=x,y,z} m_i(\mathbf{r})^2 - \sum_{\mathbf{r}} [\mu(x, y) m_z(\mathbf{r}) + H_\perp m_x(\mathbf{r})], \quad (8)$$

where we take $J = 1$, T is the temperature, and $m_i(\mathbf{r})$ is an integer describing the vorticity on the link in direction i at position $\mathbf{r}$. Tr denotes the sum over all possible configurations of integers $m_i(\mathbf{r})$ on the links of the lattice, which satisfy $\partial_i m_i = 0$, i.e., form closed loops. $\mu(x, y)$ is a random number in the interval $[0, 1]$ that models a strong magnetic field and columnar disorder in the z direction.

In the MC simulations we mainly study two quantities: the magnetic permeability μ and magnetic flux density B_x in the x direction, defined as

$$[\langle B_i \rangle] = -\frac{\partial f}{\partial H_i} = \frac{1}{L_x L_y L_z}\left[\left\langle \sum_{\mathbf{r}} m_i(\mathbf{r}) \right\rangle\right], \quad (9)$$

$$\mu_i = -\frac{\partial^2 f}{\partial H_i^2} = \frac{L_x L_y L_z}{T}[\langle B_i^2 \rangle - \langle B_i \rangle^2], \quad (10)$$

where $\langle \cdots \rangle$ and $[\cdots]$ denote the ensemble average and the disorder average, respectively. To avoid systematic errors $[\langle B_i \rangle^2]$ is calculated using two replicas of the system with the same disorder. Equations (2)–(4) imply the finite size scaling relation for μ_i of the form (similar results are obtained for B_i):

$$\mu_i(t, L_x, L, L_z) \sim L^{2p-(1+\chi+\zeta)} \tilde{\mu}_i\left(L^{1/\nu} t, \frac{L_x}{L^\chi}, \frac{L_z}{L^\zeta}\right), \quad (11)$$

where $\tilde{\mu}_i$ is a scaling function and p depends on the direction i: $p = \chi, 1, \zeta$ for $i = x, y, z$. To extract the critical exponents, we need to use system sizes of shape

$L_x \times L_y \times L_z = aL^\chi \times L \times bL^\zeta$, where a and b are constants which can be chosen arbitrarily. If χ and ζ are chosen correctly, then $\frac{TL_yL_z}{L_x}\mu_x$, $\frac{TL_xL_z}{L_y}\mu_y$, and $L_yL_zB_x$ become independent of system size at the critical point, since all the arguments in Eq. (11) are then independent of system size. χ and ζ are calculated by testing many values and checking for which combination a well defined crossing is obtained for all quantities in all directions. Note that, according to Eq. (4), μ_z is expected to be a smooth function at T_c. This is confirmed in the simulations. Note also that Eq. (6) cannot be expected to hold in detail, since the lattice in the simulated model breaks rotation invariance. However, the exponent $\chi = 2$ is still expected to hold since the critical fixed point is still expected to scale according to Eq. (5).

In order to overcome critical slowing down and to equilibrate the systems close to the transition, we use a "worm" cluster update algorithm [20,21]. We also use the parallel tempering [22] method. We start from an empty system and use 45 000 worm updates for equilibration, followed by equally many updates for measurements. We perform averages over approximately 1000–1500 disorder realizations.

Figure 2 shows the permeability scaling using the 1-2-3 scaling result. It is only when χ and ζ are very close to the 1-2-3 result that well defined crossings at the same temperature in both the x and the y directions are obtained. The critical temperature has also been determined from the magnetic flux with the same results. To test universality: Simulations have been performed for both $H_x = 0.1$ and $H_x = 0.2$. A second, weaker, disorder distribution has also been tested. Identical critical exponents, within error bars, are obtained for both values of the transverse field and for both disorder distributions.

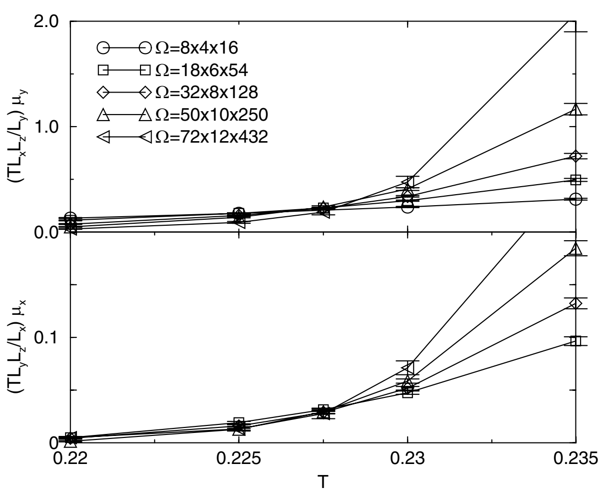

FIG. 2. MC data for the magnetic permeability scaled according to Eq. (11) in the x and the y directions for $H_\perp = 0.1$. The aspect ratio of the system sizes are chosen according to the 1-2-3 scaling. The quantity shown is equal to the winding number fluctuations in the boson language.

We proceed by using Eq. (11) to calculate ν. We note that $\frac{TL_yL_z}{L_x}\mu_x$, $\frac{TL_xL_z}{L_y}\mu_y$, or $L_yL_zB_x$ for different system sizes can be collapsed onto single curves close to T_c, if the correct value of ν and T_c are used. The values of ν and T_c are best found by a multiparameter minimization of

$$\sum_L \int (\mathcal{O}_L(x) - \bar{\mathcal{O}}(x))^2 dx, \tag{12}$$

where $\mathcal{O}_L = \frac{TL_yL_z}{L_x}\mu_x$, $\frac{TL_xL_z}{L_y}\mu_y$, or $L_yL_zB_x$, and $x = (T - T_c)L^{1/\nu}$. Here $\mathcal{O}(x)$ is obtained by linear interpolation of the data points for a given system size, and $\bar{\mathcal{O}}(x)$is its mean value. We have tested various other interpolation methods, all giving the same values for T_c and ν, within error bars. Figure 3 displays the collapse resulting from minimization of Eq. (12) with $\mathcal{O} = \frac{TL_yL_z}{L_x}\mu_x$ and $\frac{TL_xL_z}{L_y}\mu_y$ for $H_\perp = 0.1$. The best fit is obtained for $\nu = 0.7 \pm 0.1$ for both the magnetic fields that we tried ($H_\perp = 0.1, 0.2$) in both the x and the y directions. The error bars are estimated using the bootstrap method [23], by varying the range of system sizes included in the fit and by varying the temperature interval in which the fit is made. In Fig. 4 the same method is used to collapse the total flux curves. The result for T_c is within error bars the same as T_c estimated above. The value of ν is estimated to $\nu = 0.68 \pm 0.06$. This value of ν is very close to the Harris criterion bound, given above ($\nu \geq 2/3$).

Implications for experiments.—The anisotropic 1-2-3 scaling predicts a number of experimental testable results for the transverse Meissner transition. The magnetic response of the system at the transition is summarized in the scaling relations for the magnetic permeability in Eqs. (2) and (3).

We also derive scaling relations for the resistivity and the I-V characteristics. We assume that the transition is

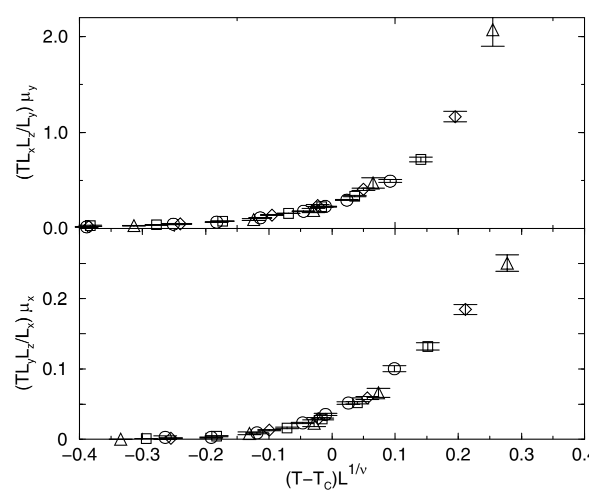

FIG. 3. Finite size scaling of MC data for the magnetic permeability for $H_\perp = 0.1$. The best collapse is obtained for $T_c = 0.228 \pm 0.003$ and $\nu = 0.7 \pm 0.1$. The quantity plotted is equal to the winding number fluctuations in the boson language.

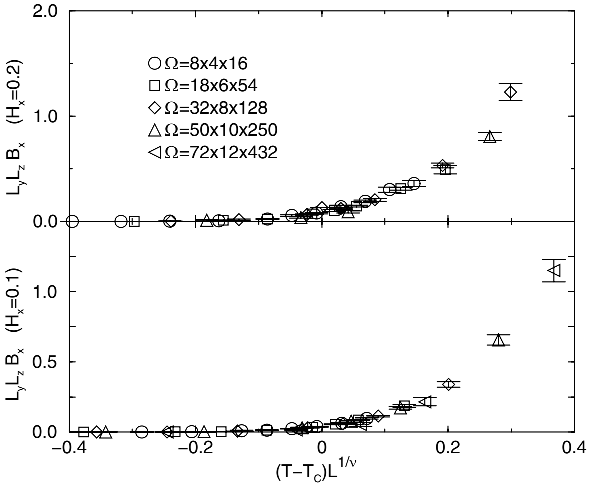

FIG. 4. Finite size scaling of MC data for the total magnetic flux in the x direction for $H_\perp = 0.1$ and $H_\perp = 0.2$. In the $H_\perp = 0.2$ case the best collapse is obtained for $T_c = 0.201 \pm 0.003$ and $\nu = 0.68 \pm 0.05$. For $H_\perp = 0.1$, $T_c = 0.227 \pm 0.003$ and $\nu = 0.67 \pm 0.06$. The quantity plotted is equal to the net winding number in the boson language.

directly from the transverse Meissner phase to the vortex liquid phase. This should be the case at small tilt angles. At larger tilt angles there are indications for a transition from a transverse Meissner phase to vortex solid [9]. The vector potential scales as $A_i \sim \xi_i^{-1}$, which follows from gauge invariance. We make the dynamic scaling ansatz for the correlation time $\tau \sim \xi^z$, where z is theoretically unknown. This implies that the electric field and the current density scale as $E_i = -\frac{1}{c}\frac{\partial A_i}{\partial t} \sim \xi_i^{-1}\xi^{-z}$, $J_i = \frac{\partial f}{\partial A_i} \sim \frac{\xi_i}{\xi_x\xi_y\xi_z}$. We thus obtain the scaling form of the electric field, $E_i\xi_i\xi^z = \tilde{E}_i(J\xi_x\xi_y\xi_z/\xi_i)$, which is different in different directions $i = x, y, z$.

Consider the linear resistivity $\rho_{ij} = \frac{\partial E_i}{\partial J_j}$. According to the scaling relations above and 1-2-3 scaling, the z independent resistivity ratios scale as $\frac{\rho_{xx}}{\rho_{zz}} \sim \xi^2 \sim (T - T_c)^{-2\nu}$, $\frac{\rho_{yy}}{\rho_{zz}} \sim \xi^4 \sim (T - T_c)^{-4\nu}$, and $\frac{\rho_{xx}}{\rho_{yy}} \sim \xi^{-2} \sim (T - T_c)^{2\nu}$. The I-V characteristics, right at the critical temperature, is predicted to be $E_x \sim J^{(z+\chi)/(\zeta+1)}$, $E_y \sim J^{(z+1)/(\zeta+\chi)}$, and $E_z \sim J^{(z+\zeta)/(1+\chi)}$ in the respective direction. Note that all these relations, which apply at finite tilt, are different from the corresponding relations for the Bose glass transition at zero tilt.

In summary, we determine the critical properties of the TM transition from exact scaling arguments, and from numerical Monte Carlo simulations. The divergence of the correlation volume at the TM transition is anisotropic in all three directions, given by $\xi^2 \times \xi \times \xi^3$, with $\xi \sim |T - T_c|^{-\nu}$, $\nu = 0.68 \pm 0.06$. This is very different from the Bose glass transition, where the correlation length is anisotropic in one direction only. Our theory makes distinct predictions that would be interesting to test experimentally on superconductors with columnar defects.

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[1] D. R. Nelson and V. M. Vinokur, Phys. Rev. Lett. **68**, 2398 (1992).
[2] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
[3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, U.K., 1999).
[4] D. R. Nelson and V. M. Vinokur, Phys. Rev. B **48**, 13 060 (1993).
[5] M. Wallin, E. S. Sørensen, S. M. Girvin, and A. Young, Phys. Rev. B **49**, 12 115 (1994).
[6] N. Hatano and D. R. Nelson, Phys. Rev. B **56**, 8651 (1997).
[7] V. T. Phuoc, E. Olive, R. D. Sousa, A. Ruyter, L. Ammor, and J. Soret, Phys. Rev. Lett. **88**, 187002 (2002).
[8] A. Tonomura *et al.*, Nature (London) **412**, 620 (2001).
[9] A. Smith, H. M. Jaeger, T. F. Rosenbaum, W. K. Kwok, and G. W. Crabtree, Phys. Rev. B **63**, 064514 (2001).
[10] A. Smith, H. M. Jaeger, T. F. Rosenbaum, A. M. Petrean, W. K. Kwok, and G. W. Crabtree, Phys. Rev. Lett. **84**, 4974 (2000).
[11] R. J. Olsson, W.-K. Kwok, L. M. Paulius, A. M. Petrean, D. J. Hofman, and G. W. Crabtree, Phys. Rev. B **65**, 104520 (2002).
[12] T. Klein, A. Conde-Gallardo, I. Joumard, J. Marcus, C. J. van der Beek, and M. Konczykowski, Phys. Rev. B **61**, R3830 (2000).
[13] S. A. Grigera, E. Morré, E. Osquiguil, C. Balseiro, G. Nieva, and F. de la Cruz, Phys. Rev. Lett. **81**, 2348 (1998).
[14] B. Maiorov and E. Osquiguil, Phys. Rev. B **64**, 052511 (2001).
[15] J. Lidmar and M. Wallin, Europhys. Lett. **47**, 494 (1999).
[16] D. R. Nelson and V. M. Vinokur, Phys. Rev. B **61**, 5917 (2000).
[17] J. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[18] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[19] A. Vestergren, J. Lidmar, and M. Wallin, Phys. Rev. B **67**, 092501 (2003).
[20] F. Alet and E. S. Sørensen, Phys. Rev. E **67**, 015701 (2003).
[21] N. V. Prokof'ev, B. V. Svistunov, and I. S. Tupitsyn, Phys. Lett. A **238**, 253 (1998).
[22] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. **65**, 1604 (1996).
[23] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C* (Cambridge University Press, Cambridge, U.K., 1992).