Uniformly Frustrated *XY* Model without a Vortex-Pattern Ordering

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The uniformly frustrated *XY* model with $f = 1/3$ on a dice lattice is shown to possess an accidental degeneracy of its ground states so well developed that the difference between the free energies of fluctuations does not lead to the stabilization of a particular vortex pattern down to zero temperature. Nonetheless, at low temperatures the system is characterized by a finite helicity modulus whose vanishing (at a finite temperature) is related to the dissociation of half-vortex pairs.

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In the presence of an external magnetic field a regular two-dimensional Josephson junction array can be described by the Hamiltonian of a uniformly frustrated *XY* model [1]:

$$H = -J \sum_{(\mathbf{jk})} \cos(\varphi_{\mathbf{k}} - \varphi_{\mathbf{j}} - A_{\mathbf{jk}}), \tag{1}$$

where J is the Josephson coupling constant, the fluctuating variables $\varphi_{\mathbf{j}}$ are the phases of the order parameter on superconducting grains forming the array, the quenched variables $A_{\mathbf{jk}}$ are determined by the vector potential of the field, and the summation is taken over all pairs of grains connected by a junction. The form of Eq. (1) assumes that the currents in the array are sufficiently small, so their proper magnetic fields can be neglected.

The directed sum of the variables $A_{\mathbf{jk}} \equiv -A_{\mathbf{kj}}$ along the perimeter of each plaquette should be equal to $2\pi f$, f being the ratio of the magnetic flux per plaquette to the flux quantum Φ_0. It is sufficient to consider the interval $f \in [0, \frac{1}{2}]$, because all other values of f can be reduced to this interval by a simple replacement of variables [1]. The case of $f = 0$ corresponds to the absence of frustration.

The most essential feature of uniformly frustrated *XY* models is the nonperturbative nature of the interaction between the discrete and continuous degrees of freedom, which is related to the formation of fractional vortices on domain walls [2]. After two decades of active investigations, a relatively complete understanding of the consequences of such an interaction has been achieved for only the models with $f = 1/2$ on square and triangular lattices, in both of which the discrete degeneracy is described by the simplest possible group, Z_2, and the loss of phase coherence is forced to take place at a lower temperature than the disordering of the vortex pattern [3]. In more complex cases (especially in the presence of an accidental degeneracy [4]) no conclusions can usually be made on the sequence of phase transitions [5], and often even the determination of a vortex-pattern structure in the low temperature phase requires substantial effort.

In the present work we demonstrate that the model with $f = 1/3$ and dice lattice (see Fig. 1) is a rare example of a uniformly frustrated *XY* model which allows for a rather detailed understanding of its behavior. Moreover, this behavior turns out to be qualitatively different from all other cases investigated up to now. Namely, in the considered model an accidental degeneracy of ground states is so well developed that the vortex pattern remains disordered at any temperature. Nonetheless, at low temperatures the helicity modulus is finite. It vanishes at the point of phase transition related to the unbinding of pairs of fractional vortices with half-integer topological charges. A possibility for the stabilization of a specific vortex pattern appears only when one goes beyond the limits of the *XY* model. In the conclusion we briefly discuss the removal of the accidental degeneracy by magnetic interactions and its consequences.

The interest in magnetically frustrated superconducting systems with a dice lattice has been motivated by the unusual properties of a single electron spectrum in this geometry [6]. In recent years superconducting wire networks and Josephson junction arrays with a dice lattice have become the subjects of active experimental investigations [7–9] and numerical simulations [10]. Magnetically frustrated Josephson junction arrays formed by rhombic plaquettes have also been discussed in the context of the creation of topologically protected qubits [11].

Since both $\varphi_{\mathbf{j}}$ and $A_{\mathbf{jk}}$ depend on a particular choice of the gauge, it is more convenient to describe different states of the system in terms of the gauge-invariant phase differences $\theta_{\mathbf{jk}} = \varphi_{\mathbf{k}} - \varphi_{\mathbf{j}} - A_{\mathbf{jk}} \equiv -\theta_{\mathbf{kj}}$, which below are

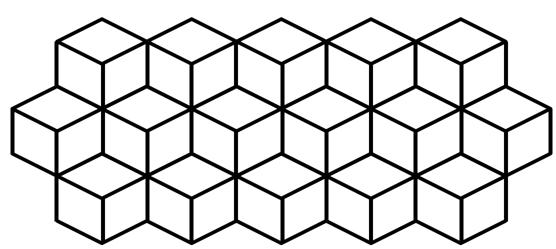

FIG. 1. The dice lattice is periodic and consists of identical rhombic plaquettes with three different orientations.

always assumed to be reduced to the interval $(-\pi, \pi)$. The subscripts **j** and **k** are used to denote sixfold and threefold coordinated sites, respectively. If one defines the variable $m_{\mathbf{jj}'}$ to be given by the directed sum of variables θ over the perimeter of the plaquette $\langle \mathbf{jkj}'\mathbf{k}' \rangle$ divided by 2π, the plaquettes for which $m_{\mathbf{jj}'}$ is equal to $1-f$ (rather then to $-f$) are usually referred to as containing vortices. Different local minima of (1) can then be classified by specifying the positions of the vortices, whose concentration for $f = 1/3$ should be exactly equal to one-third.

The remarkable property of the considered XY model is that all minima of (1) corresponding to different vortex configurations in which vortices do not occupy adjacent plaquettes have the same energy (per bond), $E_0 = -(2/3)J$. Some of these states are schematically shown in Fig. 2. In all such states the variables θ are equal to $\pi/3$ on all bonds surrounding a vortex and to zero on all other bonds (shared by two plaquettes without vortices). It is evident that these states correspond to the absolute minimum of energy, because they minimize the energy separately for each plaquette containing a vortex and for each of the remaining bonds. The barriers between the "neighboring" ground states have the height $E_b = 6(2-\sqrt{3})J \approx 1.61J$.

All these ground states [whose degeneracy survives even if the interaction in (1) deviates from cosine] can be put into correspondence with the ground states of the antiferromagnetic Ising model defined on the triangular lattice $\mathcal{T}$ formed by the sixfold coordinated sites $\{\mathbf{j}\}$. In all ground states of such a model each triangular plaquette should contain exactly one bond with parallel spins ($s_{\mathbf{j}}s_{\mathbf{j}'} = 1$) [12]. The existence of a mapping becomes evident as soon as one notices that in all ground states of the considered XY model each hexagon formed by three neighboring plaquettes should contain exactly one vortex. The plaquettes with vortices can then be identified with the bonds connecting parallel spins by setting $s_{\mathbf{j}}s_{\mathbf{j}'} = 2m_{\mathbf{jj}'} - 1/3$.

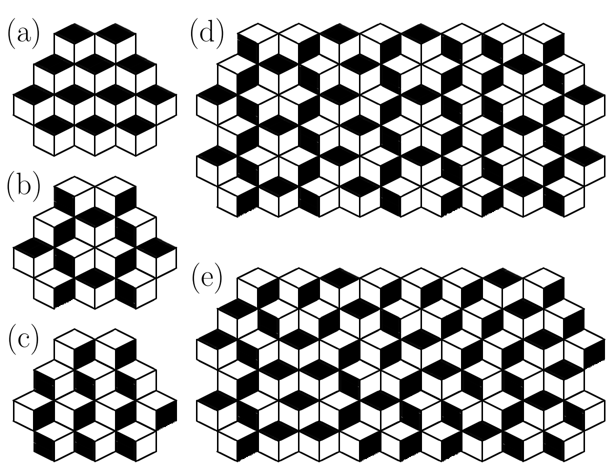

FIG. 2. Some ground states of the frustrated XY model with a dice lattice and $f = 1/3$. Filled plaquettes correspond to $m = 2/3$ and empty plaquettes to $m = -1/3$.

At zero temperature, $T = 0$, the antiferromagnetic Ising model on a triangular lattice is characterized by an algebraic decay of correlation functions [13] and a finite extensive entropy, which is related to the possibility of the creation of zero-energy domain walls forming closed loops [12]. The same partition function can be interpreted as the partition function of the SOS (solid-on-solid) model suitable for the description of the height fluctuations on the (111) facet of a crystal with a simple cubic lattice [14]. In terms of this SOS model zero-energy domain walls correspond to zero-energy steps, and the algebraic correlations of spins are translated into logarithmic correlations of integer variables $h_{\mathbf{j}}$ and $n_{\mathbf{k}}$, which can be associated with the height of the surface. These variables can be introduced following the relations

$$h_{\mathbf{j}\pm\mathbf{e}_\alpha} = h_{\mathbf{j}} \pm 3m_{\mathbf{j},\mathbf{j}\pm\mathbf{e}_\alpha}, \qquad n_{\mathbf{k}} = \frac{1}{3}\sum_{\mathbf{j}=\mathbf{j}(\mathbf{k})} h_{\mathbf{j}},$$

where $\mathbf{e}_\alpha$ (with $\alpha = 1, 2, 3$) are the three basic vectors ($\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$) of $\mathcal{T}$ and $\mathbf{j}(\mathbf{k})$ are the three nearest neighbors of $\mathbf{k}$ on the dice lattice. According to Ref. [14], for $|\mathbf{k}_1 - \mathbf{k}_2| \gg 1$,

$$\langle (n_{\mathbf{k}_1} - n_{\mathbf{k}_2})^2 \rangle \propto \frac{9}{\pi^2} \ln|\mathbf{k}_1 - \mathbf{k}_2|. \qquad (2)$$

The form of Eq. (2) demonstrates that the SOS model is in the rough phase and that at $T = 0$ the large-scale fluctuations of n can be described by a continuous Gaussian Hamiltonian,

$$H = \frac{K}{2}\int d^2\mathbf{r}(\nabla n)^2, \qquad (3)$$

where the dimensionless effective rigidity K (which is of entropic origin) is equal to $K_0 = \pi/9$ [14]. The phase transition of the SOS model to the smooth phase would take place when $K = \pi/2$ [15]; thus, the system is situated relatively far from the transition point.

The simplest periodic ground state shown in Fig. 2(a) (the striped state) has been discussed in Ref. [10]. In terms of the SOS representation this state has the maximal possible slope, whereas a flat state of the SOS model (with $n_{\mathbf{k}} = \text{const}$) corresponds to the honeycomb vortex pattern of Fig. 2(b). Figure 2(d) shows two parallel steps (of opposite signs) which separate flat states with $\Delta n = \pm 1$. If the left step of Fig. 2(d) is repeated as often as possible, one obtains the striped state of Fig. 2(a). On the other hand, the repetition of the right step of Fig. 2(d) leads to the zigzag state shown in Fig. 2(c). However, the steps do not have to be straight, and a typical ground state looks rather disordered, Fig. 2(e).

If the SOS model would be in a smooth phase, the vortex configuration would be characterized by the long-range order corresponding to the formation of the honeycomb pattern of Fig. 2(b). The fluctuations of $n_{\mathbf{k}}$ and h_{j} lead to the replacement of the true long-range order by an algebraic decay of correlations of $m_{\mathbf{j}\mathbf{j}'}$,

$$\langle m_{\mathbf{j}_1\mathbf{j}_1'} m_{\mathbf{j}_2\mathbf{j}_2'} \rangle \propto |\mathbf{j}_1 - \mathbf{j}_2|^{-\eta},$$

where $\eta = 2$, as follows from the results of Ref. [13]. However, one can expect this correlation function to be modulated according to the honeycomb pattern.

At finite temperature, $T > 0$, the equivalence between the considered XY model and the Ising model is no longer exact, because the XY model (i) allows for the existence of continuous fluctuations (spin waves) and (ii) has a more complex classification of topological defects. Numerical calculation of the integrals over the Brillouin zone which determine the free energies of harmonic fluctuations in the vicinity of the three periodic ground states shown in Figs. 2(a)–2(c) reveals that it is the lowest in the honeycomb state [16]. The difference between the free energies of fluctuations in the zigzag and honeycomb states (normalized per single site of $\mathcal{T}$) is given by γT, where $\gamma \approx 2.27 \times 10^{-3}$.

Since in terms of the SOS model the zigzag state corresponds to the sequence of steps with unit density, the same quantity can also be used as an estimate for the effective energy of a step per elementary segment, $E_{\mathrm{st}} \approx \gamma T$. The positiveness of E_{st} should lead to a decrease of fluctuations of $n_{\mathbf{k}}$, but, since one always has $E_{\mathrm{st}}/T \ll 1$, this decrease has to be relatively small. In terms of (3) the influence of a small step energy, $E_{\mathrm{st}} \approx \gamma T$, is translated into a very small ($\approx 1\%$) correction to K_0, $K = K_0 + \sqrt{3}\gamma$. This definitely leaves the system far from the transition to the smooth phase, and, therefore, cannot lead to any qualitative changes from the zero-temperature behavior.

In terms of the original XY model each step of the SOS model corresponds to a line whose crossing shifts the phase variables $\varphi_{\mathbf{j}}$ by π (with respect to what they would be in the absence of this step) on one of the three sublattices into which the triangular lattice $\mathcal{T}$ can be split. After crossing three steps of the same sign, the variables $\varphi_{\mathbf{j}}$ are shifted by π for all three sublattices. In other words, the state which is obtained in such a way differs from the original state by a global rotation of all phases by π. As a consequence, the elementary topological excitation of the considered XY model is an object where three steps of the same sign merge together (see Fig. 3) that looks like a screw dislocation with Burgers number $b = \pm 3$, on going around which the phase experiences a continuous rotation by π. For brevity we shall call such defects half-vortices. Since the phase shift by π can be achieved by a phase rotation in both directions, the signs of the two topological charges of a half-vortex (the Burgers number and the vorticity) are not related to each other and can be arbitrary.

The core of a half-vortex can be associated with the three-plaquette cluster which instead of containing exactly one vortex contains either two vortices or no vortices at all (see Fig. 3). In the framework of the antiferromagnetic Ising model the analogous defect cannot exist, because each plaquette can contain only an odd number (1 or 3) of bonds with $s_{\mathbf{j}} s_{\mathbf{j}'} = 1$. The core energy of a half-vortex should be of the order of J.

In accordance with the double nature of half-vortices their interaction consists of two contributions of different origins. Both of them are logarithmic. The first one (the direct interaction) is related to the energy which is required to create the phase twist around the cores and is completely analogous to the interaction of ordinary vortices (with integer vorticity). This interaction is characterized by the prelogarithmic factor $P_{\mathrm{V}} = (\pi/2)\Gamma$, where Γ is the helicity modulus of the system [at $T = 0$ in the honeycomb state $\Gamma = \Gamma_0 \equiv (5/4\sqrt{3})J$]. The second contribution is of entropic origin and is related to the interaction of half-vortices as dislocations. It follows from Eq. (3) that for this interaction the prelogarithmic factor is given by

$$P_{\mathrm{D}}(b) = \frac{Kb^2}{2\pi} T, \tag{4}$$

where one should put $b = 3$.

At low temperatures ($T \ll J$) the direct interaction of half-vortices is dominant, which binds them into small pairs with zero total vorticity. However, the Burgers number of such a pair does not have to be zero, but can also be equal to ± 6. This returns one to the situation in the antiferromagnetic Ising model in which the elementary topological excitations are the dislocations with $b = \pm 6$ [15,17], the only difference being a slightly larger value of K. Substitution of $K \approx K_0 = \pi/9$ and $b = 6$ in Eq. (4) gives $P_{\mathrm{D}}(6) \approx 2$, which is insufficient for such dislocations to be bound in pairs [15]. The application of the Debye-Hückel approximation to the two-dimensional Coulomb gas formed by dislocations shows that c_{D}, the concentration of free dislocations, should be exponentially small in $1/T$. The value of c_{D} determines a temperature dependent correlation radius $r_c(T) \propto c_{\mathrm{D}}^{-1/2}$, beyond which K is renormalized to zero and even the algebraic correlations of the vortex pattern are destroyed.

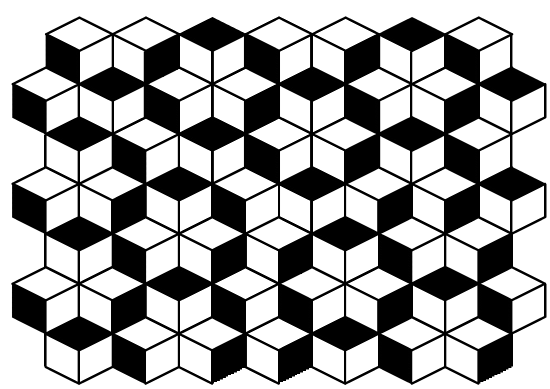

FIG. 3. A possible structure of a half-vortex.

The presence of free dislocations leads to the screening of the entropic part of the logarithmic interaction of half-vortices at the scales which are large in comparison with $r_c(T)$. Nonetheless, at low temperatures the system will be characterized by a finite value of $\Gamma(T)$, since all half-vortices will be bound in pairs by their direct interaction. With increasing temperature a phase transition will occur related to the appearance of free half-vortices and vanishing of $\Gamma(T)$. It will be completely analogous to the Berezinskii-Kosterlitz-Thouless phase transition in the conventional XY model (without frustration), the main difference being that half-vortex pairs dissociate when $T = (\pi/8)\Gamma(T)$ [18], whereas the pairs of ordinary vortices dissociate when $T = (\pi/2)\Gamma(T)$ [19].

Thus, in the present work we have shown that in the frustrated XY model with a dice lattice and $f = 1/3$ the vortex pattern is disordered at any temperature, becoming quasiordered only at $T = 0$. Nonetheless, at low temperatures the helicity modulus is finite and jumps to zero only at $T = T_{\rm HV} \sim (\pi/8)\Gamma_0 \approx 0.28J$, where the pairs of half-vortices dissociate. This estimate from above is not far from $T_c \approx 0.2J$ obtained in numerical simulations of Ref. [10].

A possibility of vortex-pattern ordering appears only when one goes beyond the limits of the XY model and takes into account some additional mechanism of the removal of an accidental degeneracy. In the case of a proximity coupled array [9] the main role will belong to the energy related to the magnetic fields of currents in the array, which is minimized in the striped state [20]. In terms of the SOS model, this leads to a negative step energy, $E_{\rm st} < 0$. In the limit of weak screening [21], when the corrections to currents from their proper magnetic fields can be neglected, $E_{\rm st} = -\mu J^2/E_\Phi$, where $E_\Phi = \Phi_0^2/4\pi^2 a$ is the characteristic energy, a is the lattice constant (of a dice lattice), and the numerical coefficient μ for both types of steps shown in Fig. 2(d) can be written as

$$\mu \approx \sin^2(\pi/3)[\lambda_2 - \lambda_4 - 2(\lambda_5 - \lambda_6) + \ldots] \approx 0.25,$$

$\lambda_i \equiv -L_i/a > 0$ being the dimensionless values of mutual inductances, L_i, between dice lattice plaquettes [22]. For $a \approx 8\ \mu$m [9], $E_\Phi \approx 10^4$ K.

With a decrease of T, the ratio $|E_{\rm st}|/T$ is increased, which at $T \gg |E_{\rm st}|$ will manifest itself only in the decrease of K in Eq. (3). With a further decrease of T, a phase transition can be expected to occur to a phase with a nonzero slope [23]. In terms of vortices this phase will be characterized by a true long-range order manifesting itself in the deviation of occupation probabilities for plaquettes with different orientations from $1/3$. An assumption that this phase transition takes place when $T \sim |E_{\rm st}|$ gives for the transition temperature an estimate, $T/J \sim (\mu T/E_\Phi)^{1/2} \sim 10^{-2}$, which is well below $T_{\rm HV}$. However, at $T/J \lesssim 0.05$ the relaxation of the vortex pattern is likely to be dynamically quenched [at $T/J = 0.05$ one has $\exp(-E_{\rm b}/T) \sim 10^{-14}$], which may prevent the observation of vortex-pattern ordering in experiments or simulations.

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