Apparent Thermalization due to Plasma Instabilities in the Quark-Gluon Plasma

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(Received 29 September 2004; published 23 February 2005)

Hydrodynamical modeling of heavy-ion collisions at RHIC suggests that the quark-gluon plasma (QGP) "thermalizes" in a remarkably short time scale, about 0.6 fm/c. We argue that this should be viewed as indicating fast isotropization, but not necessarily complete thermalization, of the nonequilibrium QGP. Non-Abelian plasma instabilities can drive local isotropization of an anisotropic QGP on a time scale which is faster than ordinary perturbative scattering processes. As a result, we argue that theoretical expectations based on weak-coupling analysis are not necessarily in conflict with hydrodynamic modeling of the early part of RHIC collisions, provided one recognizes the key role of non-Abelian plasma instabilities.

DOI: 10.1103/PhysRevLett.94.072302 PACS numbers: 25.75.Nq, 11.10.Wx, 12.38.Mh

Hydrodynamic models of Relativistic Heavy-Ion Collider (RHIC) collisions (based on near-ideal fluids) provide a good description of a wide range of experimental data, including radial and elliptic flow measurements, provided one assumes that the initial partons thermalize in about 0.6 fm/c [1]. However, theoretical estimates based on perturbative scattering processes yield expected thermalization times in the range of 2.5 fm/c or above [2]. What is the significance of this discrepancy? Are weak-coupling analyses, which should be valid for asymptotically high energy densities (and asymptotically large nuclei), inapplicable at RHIC energies? Perhaps so. Or have dynamical processes which may be responsible for this fast apparent thermalization not been correctly identified? We argue this is the case [3]. Estimates based on perturbative scattering neglect essential dynamics: the collective behavior associated with non-Abelian plasma instabilities. Such instabilities can produce large nonperturbative effects, including apparent thermalization. We discuss two qualitative lessons which emerge from a weak-coupling analysis: (i) hydrodynamic behavior does *not* require full thermalization—isotropization of parton momenta in local fluid rest frames suffices, and (ii) plasma instabilities can drive isotropization at rates which are parametrically faster than perturbative scattering rates.

Apparent thermalization.—The thermalization time scale in a quark-gluon plasma, defined as the inverse relaxation rate of arbitrarily small departures from equilibrium, depends on the rate of large-angle scattering (and near-collinear splitting or joining) processes among quarks and gluons [4]. Parametrically, this time scale is [5] $\sim 26[g^4 T \ln(2.4/g)]^{-1}$, and for plausible values of RHIC parameters it is hard to reconcile this time scale with the fast apparent thermalization observed in RHIC collisions. However, this time scale characterizing relaxation of asymptotically small perturbations is *irrelevant* to the question of when hydrodynamic models can be a good approximation to the dynamics of a nonequilibrium quark-gluon plasma. The essential assumption of ideal fluid hydrodynamic models is that the stress tensor, in the local rest frame at some point in the system, is nearly diagonal,

$$T_{ij} \approx p\delta_{ij}, \tag{1}$$

with some equation of state relating the pressure p to the energy density. But relation (1) is just a statement of *isotropy* (in the local fluid rest frame) and is automatically true if typical excitations have random directions—even if their energy distribution is far from thermal, or if the pressure p differs from the equilibrium pressure for a given energy density. Consequently, understanding when a hydrodynamic model can first provide a good approximation to the plasma dynamics is the same question as understanding what dynamics drives isotropization.

Plasma instabilities.—To begin, we summarize known results concerning gauge-field instabilities in anisotropic non-Abelian plasmas. Further details may be found in Refs. [4,6–8].

Let $p_{\rm hard}$ denote the characteristic momenta of typical excitations in a nonequilibrium quark-gluon plasma. (For example, in the saturation scenario [9], $p_{\rm hard}$ equals the saturation scale Q_s at time Q_s^{-1}.) We assume that $p_{\rm hard}$ is sufficiently large that these excitations act like highly relativistic particles. For time scales short compared to the mean free time between large-angle scatterings of typical excitations (and large compared to $p_{\rm hard}^{-1}$), the natu-

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ral framework for describing the dynamics is collisionless kinetic theory. One splits the degrees of freedom into short wavelength (or "hard" momentum) excitations which may be characterized by a phase space distribution function $f(\mathbf{p}, \mathbf{x}, t)$ and long wavelength (or "soft") gauge-field modes which may be regarded as forming a classical field. For a non-Abelian theory, the resulting Boltzmann-Vlasov equation has the form [10,11]

$$(D_t + \mathbf{v} \cdot D_{\mathbf{x}})f + \tfrac{1}{2}g\{(\mathbf{E} + \mathbf{v} \times \mathbf{B})_i, \nabla_{p_i} f\} = 0. \tag{2}$$

The corresponding Maxwell equations are

$$(D_\nu F^{\mu\nu})_a = j_a^\mu \equiv g \int_{\mathbf{p}} v^\mu \mathrm{tr}(t_a f), \tag{3}$$

with $\int_{\mathbf{p}} \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3}$, $v^\mu \equiv (1, \hat{\mathbf{p}})$, and t_a a color generator.

Any distribution which is homogeneous (in space) and colorless, combined with vanishing soft gauge field, gives a static solution to Eqs. (2) and (3). Perturbations about such solutions obey a linearized equation of motion [obtained by linearizing Eq. (2) in deviations from the static solution, solving for δf, and plugging the result into Eq. (3)] which (after a space-time Fourier transform) has the form

$$\{K^2 g^{\mu\nu} - K^\mu K^\nu + \Pi^{\mu\nu}(K)\}A_\nu(K) = 0, \tag{4}$$

where the wave vector $K^\mu \equiv (\omega, \mathbf{k})$ [12]. The retarded gauge-field self-energy, generated by hard excitations, is

$$\Pi^{\mu\nu}(K) = g^2 \int_{\mathbf{p}} \frac{\partial f(\mathbf{p})}{\partial p^l}\left[-v^\mu g^{l\nu} + \frac{v^\mu v^\nu K^l}{v \cdot K - i\epsilon}\right]. \tag{5}$$

The zero-frequency spatial self-energy $\Pi_{ij}(0, \hat{\mathbf{k}})$ depends on the direction but not the magnitude of the spatial wave vector $\mathbf{k}$. If $f(\mathbf{p})$ is anisotropic but parity invariant, then the self-energy matrix $\Pi(0, \hat{\mathbf{k}})$ has a negative eigenvalue for some directions of $\hat{\mathbf{k}}$. This implies that there are unstable solutions to the small fluctuation Eq. (4), i.e., solutions for which ω has a positive imaginary part [4,6]. These are non-Abelian versions of Weibel instabilities in ordinary plasma physics [13].

Let $-\mu^2$ denote the most negative eigenvalue of $\Pi(0, \hat{\mathbf{k}})$ (for any $\hat{\mathbf{k}}$). Unstable modes have $|\mathbf{k}| < \mu$. Let γ denote the maximal growth rate of unstable modes. If the hard particle distribution has $O(1)$ anisotropy [14], then the maximum unstable wave vector μ and the maximum growth rate γ are both comparable to the effective mass m_∞ of hard gluons,

$$\mu^2 \sim \gamma^2 \sim m_\infty^2 = g^2 \int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|}. \tag{6}$$

If p_{hard} is the momentum scale which dominates the integral (6), and $n \equiv \int_{\mathbf{p}} f(\mathbf{p})$ is the spatial density of hard excitations, then $m_\infty \sim g\sqrt{n/p_{\mathrm{hard}}}$.

To compare to perturbative scattering rates consider, for example, a system with $n = O(p_{\mathrm{hard}}^3)$—the same parametric relation as in equilibrium, where $p \sim T$ and $n = O(T^3)$. In this case m_∞, and hence the instability growth rate γ for $O(1)$ (or larger) anisotropy, is $O(g p_{\mathrm{hard}})$. This rate is parametrically faster than the $O(g^4 p_{\mathrm{hard}})$ rates for large-angle scattering or near-collinear splitting, or even the $O(g^2 p_{\mathrm{hard}})$ rate of small-angle scattering [4]. More generally, for $O(1)$ anisotropy γ is faster than the large-angle scattering rate whenever $n \ll p_{\mathrm{hard}}^3/g^2$ [15]. This inequality is satisfied parametrically unless there is saturation, and even in saturation scenarios, it is satisfied for $t \gg Q_s^{-1}$ [9].

Numerical values depend, of course, on the specific form of the anisotropic phase space distribution. A simple example [16] involving a typical particle energy of 1 GeV, plasma energy density of 27 $\mathrm{GeV/fm^3}$, a phase space distribution proportional to $(\mathbf{p} \cdot \hat{\mathbf{z}})^4$, and $\alpha_s = 0.5$ yields $m_\infty \simeq 740$ MeV, and $\gamma \simeq 280\ \mathrm{MeV} = (0.7\ \mathrm{fm}/c)^{-1}$ for $k \simeq 575$ MeV. With more extreme anisotropy, the growth rate γ can approach m_∞ itself [6]. Yet other angular distributions can give slower growth rates.

Instabilities will grow exponentially until some dynamics comes into play which causes the amplitudes of unstable modes to saturate. There are two natural possibilities for when this might happen [17]. If the unstable modes with wave numbers of order μ grow until the soft gauge field has an $O(\mu/g)$ amplitude [or the field strength is $O(\mu^2/g)$], then non-Abelian corrections to the linearized equation of motion (4) will become important and could substantially affect the further evolution [18]. In particular, one might expect these nonlinearities to lead to efficient transfer of energy from the unstable modes to stable modes (with a comparable wave number).

Alternatively, if instabilities do not saturate at $O(\mu/g)$ amplitudes, then they may continue growing until their amplitudes reach the scale p_{hard}/g [and field strengths are $O(\mu p_{\mathrm{hard}}/g)$]. This is the point where the soft gauge field no longer acts as a small perturbation on the motion of hard excitations. To see this, note that for this amplitude, the gauge-field part of a covariant derivative is just as large as the ordinary derivative when acting on fluctuations with $O(p_{\mathrm{hard}})$ momenta. This is also the point where the energy density in the soft gauge field becomes an $O(1)$ fraction of the total energy density, $(F_{\mathrm{soft}}^{\mu\nu})^2 \sim (\mu p_{\mathrm{hard}}/g)^2 \sim n p_{\mathrm{hard}}$.

There are reasons to believe the second alternative, not the first, is correct. The generalization to anisotropic plasmas of the "hard thermal loop" effective action is [19,20]

$$S_{\mathrm{eff}} = -\int d^4x \left[\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + g^2 \int_{\mathbf{p}} \frac{f(\mathbf{p})}{|\mathbf{p}|} F^a_{\alpha\mu} \left(\frac{v^\mu v^\nu}{(v \cdot D)^2}\right)_{ab} F^{b\alpha}_{\ \ \nu}\right]. \tag{7}$$

Evaluating this, explicitly, for arbitrary static fields in order to examine the corresponding effective potential is not feasible. But in the special case of fields which vary in only one spatial direction, the effective action reduces to a simple local form. Let $\hat{\mathbf{n}}$ denote the direction of the wave vector of the most unstable mode. For gauge fields which depend only on $\hat{\mathbf{n}} \cdot \mathbf{x}$, one finds that the effective potential is [21,22]

$$V[\mathbf{A}(\hat{\mathbf{n}}\cdot\mathbf{x})] = \int d^3x \left[\frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}A^a_i \Pi_{ij}(0,\hat{\mathbf{n}})A^a_j\right]. \quad (8)$$

When $\Pi(0,\hat{\mathbf{n}})$ has a negative eigenvalue this potential is unbounded below. The runaway directions of steepest-descent correspond to Abelian field configurations where the commutator terms in the field strength F^a_{ij} vanish. This suggests that non-Abelian nonlinearities may *not* cause growing instabilities to saturate at the scale μ/g, provided the field configuration evolves toward an effectively Abelian form which can continue rolling down the potential energy landscape. This behavior has been seen in time-dependent numerical simulations in $1+1$ dimensions [22,23]—the instability locally "Abelianizes" and continues growing. It is important to perform full $3+1$-dimensional simulations of the collisionless kinetic theory (2) and (3) to verify this conclusion. Such simulations are in progress [24]. Here, we assume that growth of instabilities, beyond the soft scale μ/g, will be confirmed.

Isotropization.—Growing instabilities imply that the stress tensor of the nonequilibrium system will receive growing contributions from the soft gauge field. The fastest growing linearized modes tend to decrease the anisotropy in the total stress tensor [7]. For example, if the anisotropic hard particle distribution has a prolate form, so that $T^{\rm hard}_{zz} \gg T^{\rm hard}_{xx}, T^{\rm hard}_{yy}$, then the wave vectors of the fastest growing unstable modes lie in the equatorial plane and the growth of these modes produces a soft gauge-field contribution to the stress tensor which is oblate, $T^{\rm soft}_{xx} \sim T^{\rm soft}_{yy} \gg T^{\rm soft}_{zz}$. Conversely, for an oblate hard particle distribution, the fastest growing unstable mode has its wave vector along the normal direction and generates a prolate contribution to the stress. Hence, even in the linearized regime, one can see that soft gauge-field instabilities push the system toward greater isotropy. However, the soft contribution to the stress tensor is small compared to the hard particle contribution, and the backreaction of the soft gauge field on the hard particles is a tiny perturbation, as long as the soft gauge-field amplitude is much less than $O(p_{\rm hard}/g)$.

But if the soft gauge-field amplitude reaches the scale $p_{\rm hard}/g$, then it no longer acts as a small perturbation to the dynamics of hard excitations. Recall that the radius of curvature of an excitation of momentum p and charge g in a magnetic field B is $R = p/(gB)$. If the radius of curvature is comparable to the magnetic field coherence length μ^{-1}, which means $B \sim \mu p/g$, then excitations of momentum p will undergo $O(1)$ changes in direction during traversals of any single coherence-length sized magnetic field "patch" [25].

Therefore, if unstable soft gauge-field modes with $O(\mu)$ wave vectors grow until the field strength is $O(\mu p_{\rm hard}/g)$, then typical excitations will experience $O(1)$ changes in direction in times of order μ^{-1}. Excitations with differing momenta or colors will receive different deflections from a given patch of (non-Abelian) magnetic field. Excitations traversing different patches of magnetic field [separated by $O(\mu^{-1})$] will receive nearly uncorrelated deflections.

The net effect is that a soft gauge field with a nonperturbative amplitude of order $p_{\rm hard}/g$ can effectively drive isotropization in the distribution of typical hard excitations on a time scale which equals the coherence length μ^{-1} of the soft gauge field. And isotropization of the hard particle distribution will turn off further growth in the soft gauge field (since gauge-field instabilities are absent for isotropic distributions).

As with all instabilities, the time, or number of e-foldings, required for the soft gauge field to become large depends on the size of initial "seed" amplitudes in the relevant unstable modes. The amplitude of the soft $(k \sim \mu)$ gauge field generated by a random color charge distribution of the hard particles can be estimated as $A^2 \sim g^2 n/\mu \sim g\sqrt{n p_{\rm hard}}$. This is the smallest the seed field could be. For densities from $n = O(p^3_{\rm hard})$ up to the density limit $n = O(p^3_{\rm hard}/g^2)$ imposed by saturation, $A \gtrsim O(g^{1/2} p_{\rm hard})$. This is only a factor of $g^{3/2}$ smaller than the nonperturbative $O(p_{\rm hard}/g)$ amplitude. Therefore, the number of e-foldings required for instabilities to grow to this nonperturbative size is only of order $\ln(1/g)$.

Treating logs of g, for simplicity, as $O(1)$, this means that if the initial anisotropy is $O(1)$ then the characteristic growth time needed for unstable modes of the soft gauge field to reach the nonperturbative amplitude $p_{\rm hard}/g$ is only of order γ^{-1}. The resulting soft gauge field then drives isotropization of the hard particle distribution on a comparable γ^{-1} time scale. Therefore (up to logs of g and factors of order 1), the time scale for isotropization of the hard particle distribution is the *same* as the (inverse) instability growth rate γ^{-1} [26].

In numerical simulations of ordinary nonrelativistic plasmas, essentially the same process of instability-driven isotropization has been observed [27], with the growth of magnetic instabilities driving large reductions in anisotropy once the magnetic fields reach critical strength. (These simulations allowed three-dimensional momentum space variations but assumed translation invariance in one spatial direction.) Various quark-gluon plasma numerical simulations [28] have failed to see any sign of this instability-driven dynamics because they did not allow full three-dimensional variations.

Although we have focused on the ability of nonperturbative soft gauge fields to generate large changes in directions of hard excitations, it should be noted that μ^{-1} is also the characteristic time scale for $O(1)$ changes in energies of hard excitations. This is inevitable, given the fact, noted earlier, that when the soft gauge field reaches the nonperturbative amplitude $p_{\rm hard}/g$ its energy density is comparable to the energy density in the hard excitations. But it may also be seen directly by noting that chromoelectric fields generated during the growth of instabilities will be comparable in size to chromomagnetic fields (since the growth rate of unstable modes is comparable to their wave numbers for $O(1)$ anisotropy). So chromoelectric fields will reach the same $O(\mu p_{\rm hard}/g)$ size as magnetic fields—which means that an excitation traveling a distance μ^{-1} will have work of order $p_{\rm hard}$ done on it by the soft gauge field. Of course, this time scale for $O(1)$ changes in energy may be very different (and much shorter) than the time scale for true thermalization, as defined by a near-thermal energy distribution of excitations over a parametrically large dynamic range.

Conclusions.—We have argued that "early thermalization" in heavy-ion collisions is more properly interpreted as evidence of fast isotropization in the distribution of excitations. And we have argued that non-Abelian plasma instabilities can drive isotropization at a rate which is parametrically fast compared to perturbative scattering rates. Consequently, we see no reason to view the fast onset of hydrodynamic behavior in RHIC collisions as necessarily in conflict with theoretical expectations based on weak-coupling analysis of a quark-gluon plasma, provided one properly accounts for the effects of nonperturbative plasma instabilities. Further study of the scenario we have sketched is certainly needed; in particular, full three-dimensional non-Abelian Boltzmann-Vlasov simulations with appropriate initial conditions should be conducted.

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