Effect of Phase Noise on Parametric Instabilities

F. Pétrélis, S. Aumaître, and S. Fauve

Laboratoire de Physique Statistique, Ecole Normale Supérieure, UMR 8550, 24, rue Lhomond, 75 005 Paris, France

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We report an experimental study on the effect of an external phase noise on the parametric amplification of surface waves. We observe that both the instability growth rate and the wave amplitude above the instability onset are decreased in the presence of noise. We show that all the results can be understood with a deterministic amplitude equation for the wave in which the effect of noise is just to change the forcing term. All the data for the growth rate (respectively the wave amplitude), obtained for different forcing amplitudes and different intensities of the noise, can be collapsed on a single curve using this renormalized forcing in the presence of noise.

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Introduction.—Parametric resonance occurs when the natural frequency of an oscillator or a wave is modulated in time or space. Since the early observation of parametrically amplified standing waves on the surface of a vertically vibrated layer of fluid by Faraday [1], it has been recognized that parametric resonance is involved in most areas of physics: Bragg scattering of a wave by a spatially periodic medium or by another wave, energy bands in solids, ferromagnetic resonance of spin waves, and parametric amplifiers or oscillators in electronics or optics are a few manifestations or applications of parametric resonance [2].

A problem of both theoretical and practical interest is how parametric resonance is modified when the pump, i.e., the spatial or temporal modulation, is noisy. This has been considered by Stratonovich in the context of electronic oscillators [3]. Experimental studies involve electronic oscillators [4], spin waves in ferrites and antiferromagnets [5], and parametrically amplified surface waves [6]. Although only stabilization of parametric instabilities by amplitude noise has been reported in Refs. [4,5], we have shown in [6] that the effect of noise is twofold. When the system is below the deterministic threshold, noise triggers random bursts of large amplitude oscillations, thus enhancing the instability. On the other hand, when waves are developed above the deterministic threshold, noise decreases their mean amplitude because it detunes the system away from resonance. In the spatial domain, scattering of a wave by randomly distributed scatterers instead of periodic ones, which leads to the well known problem of localization, can be also understood as an effect of noise on parametric resonance.

We study here the case when only the phase of the parametric forcing is noisy. Phase noise is of great interest because of its importance in various processes: vibratory excitation of gear systems [7], phase noise transfer in optical cavities, or limitation of the range of optical communication systems [8]. Parametric devices are phase sensitive [9]. This is used to generate squeezed states of light and microwaves [10] or to reduce thermomechanical noise [11]. However, the effect of phase noise of parametric pumping does not seem to have been studied in detail. It has been recognized that frequency modulation [12] or noise [5] inhibits parametric instabilities in ferromagnetic resonances. We show here that phase noise inhibits both the linear instability growth rate and nonlinearly saturated parametric waves, because it randomly detunes the system. We emphasize that, contrary to the case of amplitude or frequency noise, the response of the system displays a good signal to noise ratio compared to the one of the pump.

Experimental setup and measurement techniques.—The experimental setup is similar to the one of Ref. [6]. The fluid container is a plastic vessel of dimensions $95 \times 95\ \mathrm{mm}^2$ filled with mercury up to 4 mm in height. To prevent contamination of the surface, the fluid container is closed with a Plexiglas plate and mercury is kept in a nitrogen atmosphere. Its temperature is controlled by circulating water at $20.2 \pm 0.1\ ^\circ\mathrm{C}$. An electromagnetic vibration exciter, driven by a frequency synthesizer, provides a clean vertical acceleration (horizontal acceleration less than 1% of the vertical one). The vertical acceleration is measured by a piezoelectric accelerometer and a charge amplifier. The surface wave amplitude is measured by two inductive sensors (eddy-current linear displacement gauge, Electro 4953 sensors with EMD1053 DC power supply). Both sensors, 3 mm in diameter, are screwed in the Plexiglas plate perpendicularly to the fluid surface at rest. They are put 0.7 mm above the surface. The linear sensing range of the sensors allows distance measurements from the sensor head to the fluid surface up to 1.27 mm with a 7.9 V/mm sensitivity. The sensors are located on one of the diagonals of the container, 40 mm away from each other about the center. The linear response of these inductive sensors in the case of a wavy liquid metal surface has been checked in a previous study [13].

A two channel function generator provides the sinusoidal signal with adjustable phase noise, $v(t) = V \cos[\Omega t + \Phi(t)]$. We control the standard deviation of $\Phi(t)$ that is a pseudorandom Gaussian white noise. The acceleration in the reference frame of the fluid container is $g_{\rm eff} = g + a\cos[\Omega t + \Phi(t)]$, where g is the acceleration of gravity and a is proportional to V. Time recordings of the forcing and of the amplitude of the generated standing waves in the presence of noise are displayed in Fig. 1. Note that the signal to noise ratio of the response is rather high compared to the one of the pump.

Experimental results.—The instability growth rate, measured for different noise intensities, is displayed in Fig. 2. We first observe that the instability onset is delayed to larger values of critical forcing V_c with increasing noise. The growth rate varies linearly with $V - V_c$ for different noise intensities but its slope decreases with increasing noise.

The nonlinearly saturated amplitude of the standing wave generated by the parametric instability is displayed in Fig. 3 for different noise intensities. We recover that parametric waves are inhibited by phase noise. The wave amplitude increases like $(V - V_c)^{1/4}$ with or without noise (see the discussion of Fig. 5). This deserves some comment since a law of the form $(V - V_c)^{1/2}$ is observed for most supercritical bifurcations. The $(V - V_c)^{1/4}$ behavior shows that the amplitude is saturated by quintic nonlinearities, thus cubic nonlinearities vanish. This occurs in the vicinity of tricritical points for which the cubic nonlinearities change sign and the bifurcation changes from supercritical to subcritical [14]. One usually needs two control parameters to reach such points. For parametrically amplified waves, the second parameter, in addition to V, is the frequency detuning ν, i.e., the frequency difference between the eigenfrequency and half the forcing frequency, as shown in [15].

The low viscosity of our working fluid (mercury) results in a coherence length of the surface waves that is so large that their wave vector is strongly quantized [16]. In our excitation frequency range, $20 < \Omega/2\pi < 30$ Hz, the wavelength of parametrically amplified waves is roughly 10 times smaller that the size of the container, and the frequency difference between two successive resonance tongues is about 1 Hz. By tuning the excitation frequency within a 1 Hz interval, it is easy to work in the vicinity of the minimum of a resonance tongue. The wave amplitude then behaves like $(V - V_c)^{1/4}$. When the frequency is detuned from resonance, the response amplitude becomes time dependent for V only slightly above threshold, probably due to mode interaction in a system with small dissipation. Thus, no $(V - V_c)^{1/2}$ can be observed on a reasonable range of V [17].

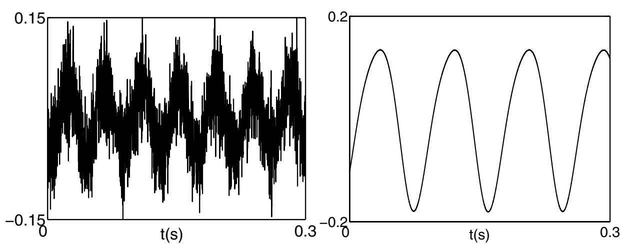

FIG. 1. Time recording of the forcing and of the response: (left) acceleration ($\mathrm{m\,s^{-2}}$) of the container with phase noise (260°). (right) Amplitude of the surface wave (mm).

Amplitude equation.—It has been shown that the amplitude $u_k(t)$, of each mode with wave number k of the surface obeys a Mathieu equation in the linear approximation for a fluid of vanishing viscosity [18],

$$\ddot{u}_k + \omega_k^2(1 + f\cos\theta)u_k = 0, \qquad (1)$$

where ω_k is the pulsation of a mode k, $f = ak/\omega_k^2$ represents the effective gravity modulation in the reference frame of the fluid container, and $\theta = \Omega t + \Phi(t)$ with $\Omega = 2(\omega_k + \nu)$. We define $a(t) = a\cos\theta(t)$. In the deterministic situation, the mode of wave number k_c, corresponding to the smallest detuning ν, is the first amplified one.

Using standard techniques [19] close to the instability onset, $f = \epsilon F$ ($\epsilon \ll 1$), we write $\nu = \epsilon\Delta$, and define a slow time scale, $T = \epsilon t$ corresponding to the instability growth rate. Expanding $v(t, T) = u_k(t)$ in power of $\sqrt{\epsilon}$, we get at leading order

$$v_0(t, T) = A(T)e^{i(\Omega/2)t} + \overline{A(T)}e^{-i(\Omega/2)t}, \qquad (2)$$

where $\overline{A}$ stands for the complex conjugate of A. The solvability condition [19] at the next order gives

$$\dot{A} = -i\Delta A + i\frac{\Omega F}{8}\langle e^{i\Phi}\rangle\overline{A}. \qquad (3)$$

Weak dissipation, $\lambda = \epsilon\Lambda$, and nonlinearities can be taken into account phenomenologically using symmetry argu-

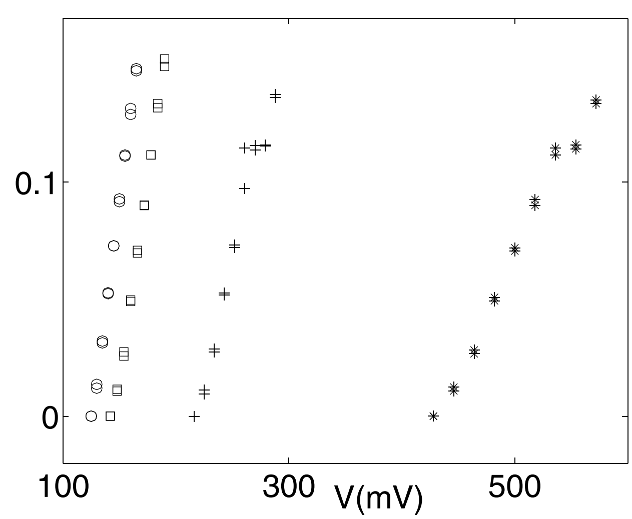

FIG. 2. Instability growth rate (s^{-1}) as a function of the forcing amplitude V with a forcing frequency $\Omega/2\pi = 23.3$ Hz, for different values of the phase noise (in degrees): (∘) 0°, (□) 86.5°, (+) 173°, (∗) 260°.

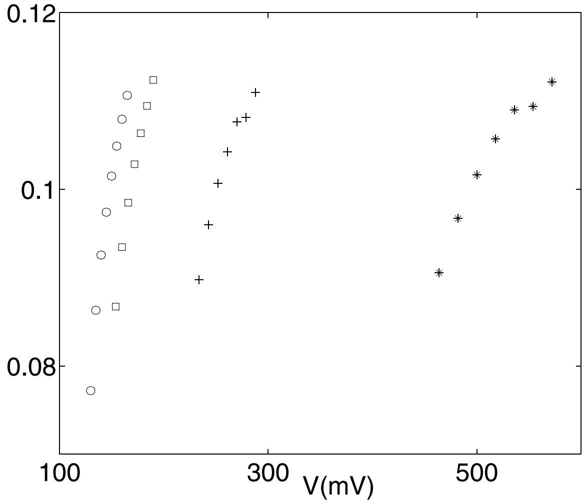

FIG. 3. Amplitude (mm) of the surface wave as a function of the forcing amplitude V with a forcing frequency $\Omega/2\pi = 23.3$ Hz, for different values of the phase noise (in degrees): (∘) 0°, (□) 86.5°, (+) 173°, (∗) 260°.

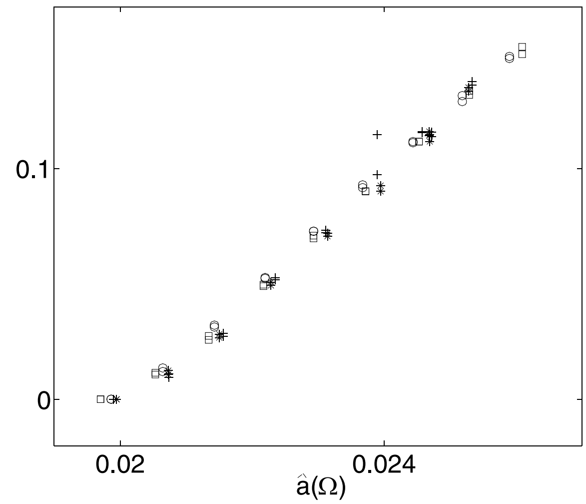

FIG. 4. Instability growth rate (s^{-1}) as a function of the Fourier component of the acceleration at pulsation Ω ($\Omega/2\pi = 23.3$ Hz), for different values of noise (in degrees): (∘) 0°, (□) 86.5°, (+) 173°, (∗) 260°.

ments [14]. For a Gaussian noise, we have $\langle e^{i\Phi}\rangle = e^{-(\langle\Phi^2\rangle/2)}$. We thus get the amplitude equation

$$\dot{A} = -(\Lambda + i\Delta)A + i\frac{\Omega F}{8}e^{-(\langle\Phi^2\rangle/2)}\overline{A} - i\beta|A|^2A, \quad (4)$$

which describes amplification and nonlinear saturation of a homogeneous standing wave or of an oscillator in the presence of parametric forcing with phase noise $\Phi(t)$. Note that Eq. (4) can be derived rigorously for a parametric pendulum. In the case of Faraday waves, additional nonlinear and damping terms may be involved (see [17]). However, whatever the exact form of the nonlinear terms, the important point here is that noise modifies the deterministic forcing through the substitution $F \to Fe^{-(\langle\Phi^2\rangle/2)}$. We first get that the instability onset given by $F_c(\Phi) = \sqrt{\Lambda^2 + \Delta^2}e^{(\langle\Phi^2\rangle/2)}$ is delayed by phase noise, as experimentally observed.

Although Eq. (4) formally involves cubic nonlinearities, for zero detuning, $\Delta = 0$, the neutral mode is saturated at quintic order in the vicinity of instability threshold [14]. We get for the nonlinearly saturated amplitude $\beta^2|A|^4 \approx 2\Lambda e^{-(\langle\Phi^2\rangle/2)}[F - F_c(\Phi)]$. In agreement with experimental results, $|A|^4$ increases linearly with the forcing amplitude above instability threshold with a slope that decreases when the noise is increased.

We observe that the Fourier component $\hat{f}(\Omega)$ of the forcing $f(t)$ at the forcing frequency Ω is proportional to $f\langle e^{i\Phi}\rangle$. In order to check the validity of the amplitude Eq. (4), we thus plot our experimental results versus $\hat{a}(\Omega) = T^{-1}|\int_0^T a(t)e^{-i\Omega t}dt|$ with $\Omega T \gg 1$. We observe that all the data, obtained for both the growth rate and the nonlinearly saturated amplitude for different noise intensities, collapse on a single curve (see Figs. 4 and 5), as predicted by Eq. (4).

Phase noise thus inhibits parametric instabilities by decreasing the amount of power of the pump available at parametric resonance. To leading order, this effect is simply taken into account in Eq. (4) for the amplitude of the unstable mode by a renormalized forcing $f \to fe^{-(\langle\Phi^2\rangle/2)}$. It should also be noted that the response of the system remains almost free of noise despite the rather noisy forcing (compare the recordings of Fig. 1). This results from the selective property of parametric amplification. The spectral part of the forcing detuned by phase noise is not amplified. All these results may *a posteriori* look obvious. A final remark shows that it is not so. If, instead of phase

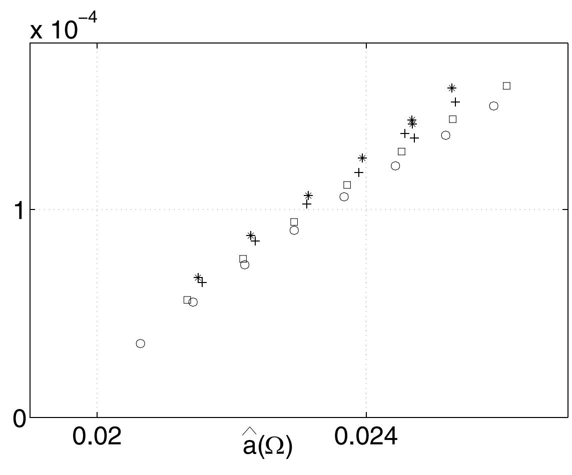

FIG. 5. Fourth power of the amplitude of the surface wave (mm^4) as a function of the Fourier component of the acceleration at pulsation Ω ($\Omega/2\pi = 23.3$ Hz), for different values of noise (in degrees): (∘) 0°, (□) 86.5°, (+) 173°, (∗) 260°.

noise, one considers the effect of frequency noise, i.e., $\dot{\theta} = \Omega + \phi(t)$ in Eq. (1), the main effect of noise is also to detune the system away from parametric resonance. However, although we still observe inhibition of the instability by frequency noise, the response does not remain free of noise. On the contrary, the fluid surface displays intermittent bursts of oscillations separated by quiet periods with a flat interface.

An extension of this work could be to consider a space dependent phase noise as in the coupled oscillator model of Ref. [20]. Although a space dependent parametric forcing is not very easy to achieve with surface waves, it is rather simple to get a noisy distribution of the local frequency of the waves using a random bottom in the shallow water limit.

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