Collective Oscillations of a Trapped Fermi Gas near the Unitary Limit

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We calculate the oscillation frequencies of trapped Fermi condensate with particular emphasis on the equation of state of the interacting Fermi system. We confirm Stringari's finding that the frequencies are independent of the interaction in the unitary limit, and we extend the theory away from that limit, where the interaction does affect the frequencies of the compressional modes only.

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The remarkable advances in producing and measuring properties of atomic condensates give a strong impetus to develop a theory to meet the challenges of interpreting the experiments. In the case of fermion condensates, one is now at the early stages of coming to an understanding of the first experimental results [1,2]. One of the characteristic properties is the frequencies of normal modes of vibration. Stringari [3] has developed the theory at the unitary limit, finding that the oscillation frequencies are independent of the details of the interaction. Here we extend the theory away from the unitary limit where the interaction has some effect.

The ''unitary limit'' is a term to describe a two-component Fermi gas with a short-range interaction, characterized by a scattering length that is large compared to the length scale set by the particle density. This limit was discussed in 1999, when one of us (G. F. B.) formulated a challenge to many-body theorists to clarify the structure of the ground state of a fictitious neutron matter, interacting with an infinite scattering length [4]. When the challenge was issued it was not really clear even if such matter is stable in principle, as either a system of bosons [5] or a system of three or more fermion species [6] is known to be unstable, due to the well-known Efimov effect. From simple dimensional arguments it was clear that the energy per particle of a two fermion species should be proportional to that of the free Fermi gas, with a universal constant of proportionality. This may be expressed as

$$\varepsilon \equiv \frac{E}{N} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\xi, \tag{1}$$

where k_F is the Fermi wave vector and ξ some pure number. Even though estimates of this number were generated at the time [7], the most convincing argument that this number is positive indeed came only recently, both in theory [8], namely $\xi \approx 0.44$, and experiments [9]. Experiments indicate so far that $\xi \approx 0.5$ with very large error bars. The theory also provides strong arguments that such a fermionic system should be superfluid at zero temperature, while the experiment is still lacking in this respect. The information inferred from various experiments so far is, at best, of a rather indirect nature and no direct unambiguous evidence of superfluidity has been reported until now. Very likely, only the direct observation of vortices will provide the ultimate experimental proof that such systems can sustain a superflow. There are strong theoretical arguments that such vortices should be almost as easily visible in the Fermi dilute atomic clouds [10] as in Bose dilute atomic clouds [11].

In the Green function Monte Carlo calculations of Refs. [8,12] it was established that the ground state energy per particle of the homogeneous phase of a system of two fermion species, interacting with a very large scattering length (much larger than the radius of the interaction r_0, namely, $|a| \gg r_0$), in the dilute regime ($nr_0^3 \ll 1$, where $n = n_\uparrow + n_\downarrow = k_F^3/3\pi^2$) is

$$\begin{aligned}\varepsilon(n) &= \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}F\left(\frac{1}{k_F a}\right) \\ &= \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\left[\xi - \frac{\zeta}{k_F a} - \frac{5\upsilon}{3k_F^2 a^2} + \mathcal{O}\left(\frac{1}{(k_F a)^3}\right)\right],\end{aligned} \tag{2}$$

where $F(x)$ is a universal function and the constants $\xi \approx 0.44$, $\zeta \approx 1$, and $\upsilon \approx 1$ (the last two values have been extracted by us from the numerical results provided by the authors [12]). We shall discuss the normal modes of oscillation for a Fermi system in a harmonic trap, whose energy per particle is given by Eq. (2). We note that the third term is independent of density and can be dropped from the present analysis.

We assume throughout that the system behaves hydrodynamically, i.e., that the pressure tensor is isotropic. If the system is superfluid, then as long as the oscillation frequency is below the gap frequency needed to break up a Cooper pair, this condition is expected to be fulfilled. The ground state density satisfies the equation

$$\nabla^2 P(n_0) + \nabla[n_0 \cdot \nabla U] = 0, \tag{3}$$

where n_0 is the ground state density, U is the trapping potential, and P is the pressure. P is related to the energy per particle $\varepsilon(n)$ by

$$P(n) = n^2\frac{d\varepsilon(n)}{dn}. \tag{4}$$

The slow and small-amplitude normal modes satisfy the following equation

$$-m\nabla^2[c^2(n_0)n_1] - \nabla(n_1 \cdot \nabla U) = m\omega^2 n_1, \qquad (5)$$

where n_1 is the oscillating density and

$$c^2(n) = \frac{1}{m}\frac{dP(n)}{dn} \qquad (6)$$

gives the speed of sound c in a uniform condensate at density n. It is straightforward to show that Kohn's generalized theorem [13] (stating that the frequencies of the dipole modes are the trap frequencies) is satisfied. It is convenient to make a change of variable in this equation, replacing n_1 by the variable f_1 defined by

$$n_1 = n_0 f_1/c^2(n_0). \qquad (7)$$

After some algebra making use of Eq. (3), we find

$$\nabla \cdot (n_0 \nabla f_1) = -\omega^2 \frac{n_0}{c^2(n_0)} f_1. \qquad (8)$$

This equation has the formal advantage of being Hermitian, and thus easier to study in perturbation theory.

As noted before [14], the equations of motion admit simple scaling solutions for gases in harmonic traps that satisfy polytropic equations of state. Before discussing a perturbative treatment, we shall present results of an analysis under the assumption of a polytropic equation of state. As we were completing this work, we learned of a similar analysis by Heiselberg, who presents a general solution in terms of hypergeometric functions (which reduce to polynomials in this case) [15]. The polytropic equation of state is

$$P(n) = \alpha n^\gamma, \qquad (9)$$

where the constant γ is the adiabatic index of the gas. The ground state density is given by

$$n_0(\mathbf{r}) = \left[\frac{\gamma - 1}{\gamma\alpha}\mu(\mathbf{r})\right]^\nu. \qquad (10)$$

Here we introduced

$$\mu(\mathbf{r}) = \mu_0 - U(\mathbf{r}), \qquad \gamma = 1 + \frac{1}{\nu}, \qquad (11)$$

where μ_0 is the chemical potential and the harmonic trap potential is given by

$$U(\mathbf{r}) = \frac{m\omega_0^2(x^2 + y^2 + \lambda^2 z^2)}{2}. \qquad (12)$$

The corresponding local sound speed is given by

$$c^2(\mathbf{r}) = \frac{\mu(\mathbf{r})}{\nu m}. \qquad (13)$$

Using these expressions, Eq. (8) can be written in the form

$$\frac{\mu(\mathbf{r})}{\nu m}\Delta f_1 + \nabla\frac{\mu(\mathbf{r})}{m} \cdot \nabla f_1 = -\omega^2 f_1. \qquad (14)$$

Because of the particular polynomial form of $\mu(r)$ the eigenfunctions f_1 have a polynomial character as well. From Eq. (2) one can extract an effective adiabatic index for a Fermi gas in the vicinity of a Feshbach resonance

$$\gamma = \frac{d\ln P}{d\ln n} = \frac{5}{3}\left[1 + \frac{\zeta}{10\xi k_F a} + \mathcal{O}\left(\frac{1}{(k_F a)^2}\right)\right]. \qquad (15)$$

Note that the quadrupole mode in the spherical condensate and the transverse quadrupole modes in the deformed condensate do not depend on the equation of state. That is because the flow in these modes is incompressible and the internal energy does not change during the oscillation cycle. The frequencies of the monopole and of the two compressional modes in the deformed condensate have a dependence on γ, and we can use that to estimate the frequency shift; see Table I.

Equation (15) shows that the effective adiabatic index is larger than $5/3$ on the Bose-Einstein condensation (BEC) side of the Feshbach resonance (when $a > 0$). This behavior implies that the frequency of the radial oscillations should increase as well when going from the BCS to the BEC side of the Feshbach resonance. This conclusion agrees with the conjecture made by Stringari [3], except that now the frequency shift has been evaluated explicitly in terms of the properties of the system. In the unitary limit, when $\gamma = 5/3$, these results also agree with previous results for the noninteracting Fermi systems in traps [16] and the results for a superfluid Fermi system away from a Feshbach resonance in a spherical trap [17]. The quadrupole frequencies obtained using scaling solutions [14] and the sum-rule approach [18], in the limit of a noninteracting Fermi gas, are different, and indeed correspond to the diabatic limit or collisionless regime [19]. In this limit the sphericity of the Fermi surface is lost during oscillations and the cloud behaves like a normal Fermi gas in Landau's zero sound regime.

The polytropic analysis is useful to show the basic dependence on the system parameters, but the parameter k_F is ill-determined, due to the nonuniformity of the condensate. We shall therefore use perturbation theory to make

TABLE I. Results for a polytropic gas. For $\lambda \ll 1$ only leading terms are shown and $c_{1,2,3}$ are some constants.

Trap type	Mode	f_1	ω^2/ω_0^2
Spherical	$L = 1$	x, y, z	1
$\lambda = 1$	$L = 2$	xy, etc.	2
	$L = 0$	$1 - 2r^2$	$3\gamma - 1$
Axial	$M = \pm 2$	$xy, x^2 - y^2$	2
$\lambda \ll 1$	$M = \pm 1$	xz, yz	$1 + \lambda^2$
	Radial	$x^2 + y^2 + c_1\lambda^2 z^2 + c_2$	2γ
	Axial	$z^2 + c_3$	$\lambda^2(3\gamma - 1)/\gamma$

a more quantitative analysis. Since Eq. (8) is Hermitian, the following formula is variational in the f_1,

$$\omega^2 = \frac{\int |\nabla f_1|^2 n_0 d^3 r}{\int f_1^2 \frac{n_0}{c^2} d^3 r}. \quad (16)$$

This implies that one can ignore the perturbation in f_1 in calculating the first-order shift in the frequency. That shift is then given by

$$\frac{\delta\omega^2}{\omega^2} = \frac{\int |\nabla f_1|^2 \delta n d^3 r}{\int |\nabla f_1|^2 n_0 d^3 r} - \frac{\int f_1^2 \delta(\frac{n}{c^2}) d^3 r}{\int f_1^2 \frac{n_0}{c^2} d^3 r}. \quad (17)$$

To apply this formula, we take the unperturbed densities n_0 and n_0/c^2 from Eqs. (10) and (13), setting $\gamma = 5/3$. The needed functions are

$$n_0 = \frac{1}{3\pi^2}\left(\frac{2m\mu}{\xi\hbar^2}\right)^{3/2}, \qquad c^2 = \frac{2\mu}{3m}. \quad (18)$$

We now have to determine the change in n and n/c^2 induced by the second term in the energy function equation (2). Including that term, the chemical potential satisfies the equation

$$\mu_0 = \varepsilon + n\frac{d\varepsilon}{dn} + U = \frac{\hbar^2 k_F^2}{2m}\xi - \frac{2\hbar^2 k_F}{5ma}\zeta + U. \quad (19)$$

We may use this expression to evaluate the first-order change in ground state number density n_0. Since the frequency does not depend on μ_0, we may hold it fixed in doing the variation. The result is

$$\delta n = \frac{4}{5\pi^2}\frac{m\zeta}{\hbar^2\xi^2 a}\mu. \quad (20)$$

In the same way, we include the perturbation in the formula for c^2 and obtain that the corresponding first-order change in n/c^2 is given by

$$\delta\left(\frac{n}{c^2}\right) = \frac{4}{5\pi^2}\frac{m^2\zeta}{\hbar^2\xi^2 a}. \quad (21)$$

We next evaluate the integrals in Eq. (17). It is convenient to change lengths to a dimensionless form, scaling them by the transverse condensate radius $R = \sqrt{2\mu_0/m\omega_0^2}$,

$$R(\tilde{x}, \tilde{y}, \tilde{z}) = (x, y, \lambda z). \quad (22)$$

In terms of the scaled variable $\tilde{\mu} = 1 - \tilde{r}^2$, Eq. (16) for the unperturbed frequency may be expressed as

$$\omega^2 = \frac{\omega_0^2}{3}\frac{\int |\tilde{\nabla} f_1|^2 \tilde{\mu}^{3/2} d^3\tilde{r}}{\int f_1^2 \tilde{\mu}^{1/2} d^3\tilde{r}}. \quad (23)$$

Equation (17) for the frequency shift becomes

$$\frac{\delta\omega^2}{\omega^2} = \frac{\zeta}{\xi^{1/2}}\frac{\hbar}{m\omega_0 Ra}K = \frac{\zeta}{\xi}\frac{1}{k_F(0)a}K, \quad (24)$$

where the dimensionless factor K is given by

$$K = \frac{6\int |\nabla f_1|^2 \tilde{\mu} d^3\tilde{r}}{5\int |\nabla f_1|^2 \tilde{\mu}^{3/2} d^3\tilde{r}} - \frac{4\int f_1^2 d^3\tilde{r}}{5\int f_1^2 \tilde{\mu}^{1/2} d^3\tilde{r}}, \quad (25)$$

and $k_F(0)$ is the value of the local Fermi momentum at the center of the trap. The prefactor in Eq. (24) displays the scaling of the frequency shift with respect to the physical parameters of the condensate. As expected, the shift is inversely proportional to the combination $k_F a$. Finally, it has a nontrivial dependence on ξ and ζ, the universal parameters defining the energy per particle in the vicinity of a Feshbach resonance.

All the needed radial integrals have the form

$$I_{m,n} = \int_0^1 \tilde{r}^m (1 - \tilde{r}^2)^n d\tilde{r}, \quad (26)$$

$$J_{m,n} = \int_0^1 \tilde{r}^m (1 - \tilde{r}^2)^n (1 - b\tilde{r}^2 + c\tilde{r}^4) d\tilde{r}, \quad (27)$$

after integration over angular variables. Since in Eq. (25) both denominators and numerators have the same angular dependence, the specific values of the angular integrals cancel out in the case of incompressible modes. We present the results for the various cases in Table II; see also Ref. [20]. One sees that the shift of the dipole mode vanishes, as required by the generalized Kohn theorem. The shift also vanishes for the pure quadrupole modes, for reasons noted earlier. The cases of most interest are the monopole mode in spherical traps ($\lambda = 1$) and the $M = 0$ modes in axially deformed (essentially cylindrical) traps with $\lambda \ll 1$. The K factors are nonvanishing in these cases, but they are rather small; for example, $K_{\mathrm{radial}} \approx 0.124$. This has the same order of magnitude as the factor $1/10$ in Eq. (15) for the change in the effective adiabatic index. The 25% difference between these two results are, however, significantly larger than the attainable experimental accuracy. We also note that the sum of factors K determining the shifts for the radial and axial modes equals the factor K for the pure monopole mode in the spherical case.

The two experimental results available so far for the frequency shifts of the radial modes [1,2] are still in noticeable disagreement with each other to permit a detailed comparison with theory and so far it is unclear why these two experiments disagree. Nevertheless, both experiments show distinctly a qualitative agreement with theory

TABLE II. Results for K.

Trap type	Mode	f_1	ω	K
Spherical	Dipole	z	ω_0	0
$\lambda = 1$	Monopole	$R^2 - 2r^2$	$2\omega_0$	$\frac{256}{525\pi}$
	Quadrupole	xy	$\sqrt{2}\omega_0$	0
Axial	$M = \pm 2$	$xy, x^2 - y^2$	$\sqrt{2}\omega_0$	0
$\lambda \ll 1$	$M = \pm 1$	$xz,\ yz$	ω_0	0
	Radial	$x^2 + y^2 + \frac{2}{5}\lambda^2 z^2 - \frac{2}{5}R^2$	$\sqrt{10/3}\omega_0$	$\frac{1024}{2625\pi}$
	Axial	$R^2 - 6\lambda^2 z^2$	$\sqrt{12/5}\lambda\omega_0$	$\frac{256}{2625\pi}$

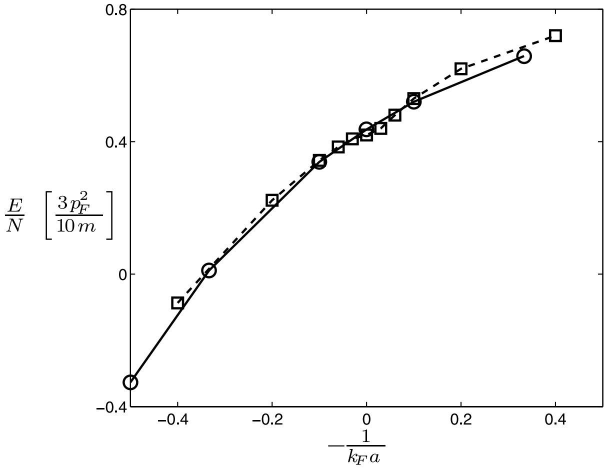

FIG. 1. A comparison between the results of Ref. [12] (solid line with open circles) and of Ref. [24] (dashed line with squares) for the energy per particle as a function of the $1/k_F a$.

as far as the character of the frequency shift is concerned, in the vicinity of the Feshbach resonance. The fact that both experiments seem to favor the adiabatic character of the oscillations should not be interpreted yet as a confirmation of the existence of superfluidity in these systems, since the sphericity of the Fermi surface can be maintained by collisions. In this respect the situation here is to some extent similar to the expansion of a cold Fermi gas [21,22]. It is also important to determine experimentally the frequencies of the transversal quadrupole modes, since a shift in these frequencies can point to a complex structure of the cloud, similar to that discussed in Ref. [23].

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Note added.—After submitting this manuscript we learned of a few other studies of Fermi systems using either the polytropic equation of state [25] or more sophisticated approaches [26]. A new calculation of the equation of state of a dilute Fermi gas was published by Astrakharchik *et al.* [24]. A fit of the results of Ref. [24] for the total energy per particle in the region $1/|k_F a| \leq 0.5$ leads to values of the parameters $\xi \approx 0.44$ and $\zeta \approx 1$, in agreement with the results of Ref. [12]; see Fig. 1.

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