Extended Hubbard Model with Ring Exchange: A Route to a Non-Abelian Topological Phase

Michael Freedman,[1] Chetan Nayak,[1,2] and Kirill Shtengel[1]

[1]*Microsoft Research, One Microsoft Way, Redmond, Washington 98052, USA*

[2]*Department of Physics and Astronomy, University of California, Los Angeles, California 90095-1547, USA*

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We propose an extended Hubbard model on a 2D kagomé lattice with an additional ring exchange term. The particles can be either bosons or spinless fermions. We analyze the model at the special filling fraction $1/6$, where it is closely related to the quantum dimer model. We show how to arrive at an exactly soluble point whose ground state is the ''d-isotopy'' transition point into a stable phase with a certain type of non-Abelian topological order. Near the ''special'' values, $d = 2\cos\pi/(k+2)$, this topological phase has anyonic excitations closely related to SU(2) Chern-Simons theory at level k.

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Since the discovery of the fractional quantum Hall effect in 1982 [1], topological phases of electrons have been a subject of great interest. Many Abelian topological phases have been discovered in the context of the quantum Hall regime [2]. More recently, high-temperature superconductivity [3–9] and other complex materials have provided the impetus for further theoretical studies of and experimental searches for Abelian topological phases. The types of microscopic models admitting such phases are now better understood [10–12].

Much less is known about non-Abelian topological phases, apart from some tantalizing hints that the quantum Hall plateau observed at $\nu = 5/2$ might correspond to such a non-Abelian phase [13–16]. However, non-Abelian topological states, if created and controlled, would open the door to scalable quantum computation [17,18]. As a first step, the study of a class of topological field theories has been reduced to combinatorial manipulations of loops on a surface [19,20]. A virtue of this formulation is that it exposes a strategy for constructing microscopic physical models which admit the corresponding phases; since Hilbert space is reduced to a set of pictorial rules, the models should impose these rules as energetically favorable conditions satisfied by the ground state (GS). In this Letter, we show how this approach can be implemented.

We propose a microscopic model which has the following properties: (a) it is an extension of the Hubbard model and, therefore, is quasirealistic, (b) it is soluble, and (c) for certain model parameters, it is perched at a transition point [21] into a non-Abelian topological phase relevant to quantum computation. By quasirealistic, we mean that the model has short-ranged interactions and hopping, so it is possible that the Hamiltonian of a real material could be viewed as a small perturbation of our Hamiltonian. Optical lattices [22], quantum dot arrays, or Josephson junction arrays [23] might be designed with Hamiltonians in this general class and thus can be promising avenues for realizing our model.

The aforementioned non-Abelian topological phases are related to the doubled $SU(2)_k$ Chern-Simons theories [20,24]. These phases are characterized by $(k+1)^2$-fold GS degeneracy on a torus and should be viewed as a natural family containing the topological (deconfined) phase of Z_2 gauge theory as its initial element, $k = 1$. For $k \geq 2$ the excitations are non-Abelian. For $k = 3$ and $k \geq 5$ the excitations are computationally universal [25]. Here, we describe the conditions which a microscopic model should satisfy to be in such a topological phase. It is useful to think of such a microscopic model as a lattice regularization of a continuum model whose low-energy Hilbert space may be described as a quantum loop gas. More precisely, a state is defined as a collection of nonintersecting loops [21,24,26]. A Hamiltonian acting on it can do the following: (i) the loops can be continuously deformed—we call this an isotopy move; (ii) a small loop can be created or annihilated—the combined effect of this move and the isotopy move has been dubbed ''d isotopy'' [20,21,24]; (iii) finally, when exactly $k+1$ strands come together in some local neighborhood, the Hamiltonian can cut them and reconnect the resulting ''loose ends'' pairwise so that the newly formed loops are still nonintersecting. More specifically, in order for this model to be in a topological phase, the GS of this Hamiltonian should be a superposition of all such pictures with the additional requirements that (i) if two pictures can be continuously deformed into each other, their amplitudes are equal; (ii) the amplitude of a picture with an additional loop is d times that of a picture without such a loop; (iii) this superposition is annihilated by the application of the Jones-Wenzl (JW) projector that acts locally by reconnecting $k+1$ strands. Readers interested in the details are referred to [20,24,26] and references therein. In this Letter, we focus on the first two conditions, which place the system at a transition point into the desired phase(s) [21]; our purpose is to construct a Hamiltonian which enforces d isotopy for its ground state(s).

Our proposed model is defined on the kagomé lattice shown in Fig. 1. The sites of the lattice are not completely equivalent; in particular, we choose two special sublattices—$\mathcal{R}$ (red) and $\mathcal{G}$ (green) whose significance will be discussed later. The Hamiltonian is given by

$$H = \sum_i \mu_i n_i + U_0 \sum_i n_i^2 + U \sum_{(i,j)\in\bigcirc} n_i n_j + \sum_{((i,j))\notin\bigcirc} V_{ij} n_i n_j - \sum_{\langle i,j\rangle} t_{ij}(c_i^\dagger c_j + c_j^\dagger c_i) + T_R. \quad (1)$$

Here $n_i \equiv c_i^\dagger c_i$ is the occupation number on site i, and μ_i is the corresponding chemical potential. U_0 is the usual on-site Hubbard energy (clearly superfluous for spinless fermions). U is a (positive) Coulomb penalty for having two particles on the same hexagon while V_{ij} represent a penalty for two particles occupying the opposite corners of "bow ties" (in other words, being next-nearest neighbors on one of the straight lines). We do not assume that all V_{ij} are equal. Specifically, $V_{ij} = v_{ab}^c$ where a is the color of site (i), b is that of (j), and c is the color of the site between them. So for the lattice in Fig. 1 we have v_{bb}^g, v_{bb}^b, v_{bg}^g, v_{rb}^b, and v_{rg}^b, where $r \in \mathcal{R}$, $g \in \mathcal{G}$, and $b \in \mathcal{B} = \mathcal{K}\backslash(\mathcal{R} \cup \mathcal{G})$. The nearest-neighbor tunneling amplitude t_{ij} is also assumed to depend on the color of the environment: $t_{ij} \equiv t_{ab}^c$ where c now refers to the color of the third site in a triangle. Finally, we include T_R—a four-particle ring exchange term whose exact form will be specified later. T_R is added to the Hamiltonian on an *ad hoc* basis to allow correlated multiparticle hops. Ring exchange terms can be justified semiclassically [27], and they do, indeed, appear in such physical systems as spin systems [28,29], solid ^{3}He [30], and Wigner crystals [31]. Of course, small ring terms can arise perturbatively along the lines of [29]; e.g., a four-particle move occurs at order 4.

The on-site Hubbard energy U_0 is assumed to be the biggest energy in the problem, and we shall set it to infinity, thereby restricting our attention to the low-energy manifold with sites either unoccupied or singly occupied. The rest of the energies satisfy the following relations: $U \gg t_{ij}, V_{ij}, \mu_i$; we shall be more specific about relations between various t_{ij}'s, V_{ij}'s and μ_i's later.

The "undoped" system corresponds to the filling fraction 1/6. The lowest-energy band then consists of configurations in which there is exactly one particle per hexagon; hence, all U terms vanish. These states are easier to visualize if we consider a triangular lattice $\mathcal{T}$ whose sites coincide with the centers of hexagons of $\mathcal{K}$. ($\mathcal{K}$ is a *surrounding* lattice for $\mathcal{T}$.) Then a particle on $\mathcal{K}$ is represented by a dimer on $\mathcal{T}$ connecting the centers of two adjacent hexagons of $\mathcal{K}$. The condition of one particle per hexagon translates into the requirement that no dimers share a site. In the 1/6-filled case this low-energy manifold coincides with the set of all dimer coverings (perfect matchings) of $\mathcal{T}$. The "red" bonds of $\mathcal{T}$ (the ones corresponding to the sublattice $\mathcal{R}$) themselves form one such dimer covering, a so-called "staggered configuration, which contains no "flippable plaquettes," or rhombi with two opposing sides occupied by dimers (see Fig. 1).

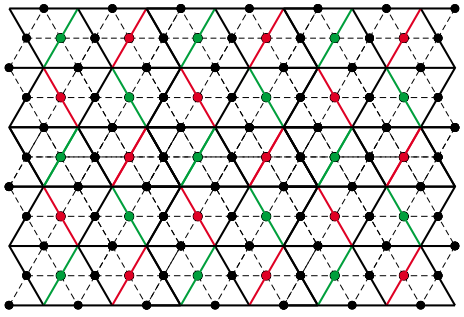

FIG. 1 (color). Solid dots and dashed lines represent sites and bonds of the kagomé lattice $\mathcal{K}$ with the special sublattices $\mathcal{R}$ (red) and $\mathcal{G}$ (green). $\mathcal{K}$ is a surrounding lattice for a triangular lattice $\mathcal{T}$ (solid lines).

So, henceforth, particles live on *bonds* of the triangular lattice (Fig. 1) and are represented as dimers [32]. In particular, a particle hop corresponds to a dimer "pivoting" by 60° around one of its end points, and $V_{ij} = v_{ab}^c$ is now a potential energy of two parallel dimers on two opposite sides of a rhombus (with c being the color of its short diagonal). It is clear that our model is in the same family as the quantum dimer model [4], which has recently been shown to have an Abelian topological phase ($k = 1$, $d = 1$) on the triangular lattice [10]. Here, we show how other values of d can be obtained.

The goal now is to derive the effective Hamiltonian acting on this low-energy manifold represented by all possible dimer coverings of $\mathcal{T}$. Our analysis is perturbative in $t/U =: \epsilon$. The initial, unperturbed GS manifold for $U_0 = \infty$, U large and positive, all $t_{ij}, V_{ij} = 0$, and all μ_i equal is spanned by the dimerizations $\mathcal{D}$ of the triangular lattice $\mathcal{T}$. As we gradually turn on the t's, v's, and T_R, we shall see what equations they should satisfy so that the effective Hamiltonian on $\mathcal{D}$ has the desired d-isotopy space as its ground state(s).

Since a single tunneling event in $\mathcal{D}$ always leads to "collisions" (two dimers sharing an end point) with energy penalty U, the lowest order at which the tunneling processes contribute to the effective low-energy Hamiltonian is 2. At this order, the tunneling term leads to two-dimer "plaquette flips" as well as renormalization of bare on-site potentials μ_i due to dimers pivoting out of their positions and back. The bare potentials μ_i are chosen to maintain equality among the renormalized $\tilde{\mu}_i$'s. This freedom to engineer the chemical potential landscape is essential to finding an exactly soluble point.

Let us pause and discuss the connection between our quantum dimer model and a desired topological phase. It is an old idea (see e.g. [33]) to turn a dimerization (perfect matching) J into a collection of loops by using a background dimerization $\mathcal{R}$ to form a transition graph $\mathcal{R} \cup J$. It turns out that fixing the preferred background dimerization $\mathcal{R}$ as in Fig. 1, we obtain the fewest equations and also achieve ergodicity [34] under a small set of moves. Unlike in the usual case, the background dimerization $\mathcal{R}$ is not merely a guide for the eyes, it is *physically* distinguished: the chemical potentials and tunnelling amplitudes are different for bonds of different color.

Let us list here the elementary dimer moves that preserve the proper dimer covering condition: (i) Plaquette (rhombus) flip—a two-dimer move around a rhombus made of two lattice triangles. Depending on whether a red bond forms a side of such a rhombus, its diagonal, or is not found there at all, the plaquettes are referred to, respectively, as type 1 (or $1'$), 2, or 3 (see Fig. 2). The distinction between plaquettes of type 1 and $1'$ is purely directional; it is necessary since our Hamiltonian breaks the rotational symmetry of a triangular (or kagomé) lattice. (ii) Triangle move—a three-dimer move around a triangle made of four elementary triangles, e.g., 4 in Fig. 2. (iii) Bow tie move—a four-dimer move around a "bow tie" made of six elementary triangles, e.g., 5 in Fig. 2.

To make each of these moves possible, the actual dimers and unoccupied bonds should alternate around a corresponding shape. Notice that for both triangle and bow tie moves we chose to depict the cases with the maximal possible number of participating red bonds (two and four, respectively). Note that there are no alternating red or black rings of fewer than eight bonds (occupied by at most four noncolliding dimers). Ring moves occur only when red bonds and dimers alternate; the triangle labeled 4 in Fig. 2 does not have a ring exchange term associated with it, but the bow tie labeled 5 does:

$$T_{\mathrm{R}} = a \left| \lambda \; - \; \right\rangle \left\langle \lambda \; - \; \right| . \qquad (2)$$

Here is the correspondence between the previous smooth discussion and rhombus flips relating dimerizations J of $\mathcal{T}$. Our surface is now a planar domain with, possibly, periodic boundary conditions (a torus). A collection of loops is generated by $\mathcal{R} \cup J$ (with the convention that the dimers of $\mathcal{R} \cap J$ be considered as length 2 loops or bigons). What about isotopy? Move 2 certainly is an isotopy from $\mathcal{R} \cup J$ to $\mathcal{R} \cup J'$, but by itself, it does almost nothing. It is impossible to build up large moves from type 2 alone. So it is a peculiarity of the rhombus flips that we have no good analog of isotopy alone but instead go directly to d isotopy. We should impose the following relations associated with moves of type 1 ($1'$):

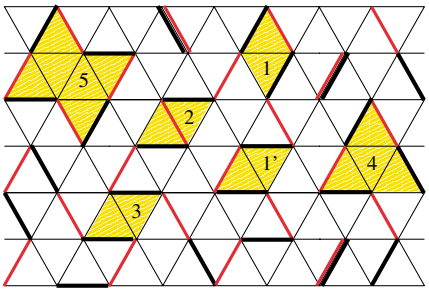

FIG. 2 (color). Overlap of a dimer covering of $\mathcal{T}$ (shown in thick black) with the red covering corresponding to the special sublattice $\mathcal{R}$. Shaded plaquettes correspond to various dimer moves described in the text. Green sublattice is not shown.

$$d\,\Psi\left(\right) - \Psi\left(\right) = 0 \qquad (3)$$

since we pass from zero to one loop in (3). Additionally, the ring exchange term (2) annihilates the superposition of one and four loops; we therefore require that $\lambda = d^3$.

Having stated our goal, we now derive the effective Hamiltonian $\tilde{H}: \mathcal{D} \to \mathcal{D}$ on the span of dimerizations. The derivation is perturbative to the second order in ϵ where $\epsilon = t^r_{bb}/U = t^b_{gb}/U$. Additionally, $t^b_{rb}/U = c_0\epsilon$ where c_0 is a positive constant, while $t^g_{bb} = o(\epsilon)$ and can be neglected in the second-order calculations. (In the absence of a magnetic field all t's can be made real and hence symmetric with respect to their lower indices. Also, we set $U = 1$ for notational convenience.) We account for all second-order processes taking us out of $\mathcal{D}$ and then back to $\mathcal{D}$. These amount to off-diagonal (hopping) processes—"plaquette flips" or "rhombus moves"—as well as diagonal ones in which a dimer pivots out and then back into its original position. The latter processes renormalize the bare on-site potentials μ_i, which we should adjust to keep all *renormalized* potentials $\tilde{\mu}_i$ equal up to corrections $\mathcal{O}(\epsilon^3)$. The effective Hamiltonian then comes from the former processes and can be written in the form: $\tilde{H} = \sum_{I,J} (\tilde{H}_{IJ} \otimes \mathbb{1}) \Delta_{IJ}$ where $\tilde{H}_{IJ}$ is a 2×2 matrix corresponding to a dimer move in the two-dimensional basis of dimer configurations connected by this move. $\Delta_{IJ} = 1$ if the dimerizations $I, J \in \mathcal{D}$ are connected by an allowed move, $\Delta_{IJ} = 0$ otherwise. Therefore, it suffices to specify these 2×2 matrices $\tilde{H}_{IJ}$ for the off-diagonal processes. For moves of types (1)–(3), they are given below:

$$\tilde{H}^{(1)} = \begin{pmatrix} v^b_{gb} & -2t^b_{rb}t^b_{gb} \\ -2t^b_{rb}t^b_{gb} & v^b_{rb} \end{pmatrix} = \begin{pmatrix} v^b_{gb} & -2c_0\epsilon^2 \\ -2c_0\epsilon^2 & v^b_{rb} \end{pmatrix}, \qquad (4a)$$

$$\tilde{H}^{(1')} = \begin{pmatrix} v^b_{bb} & -2t^b_{rb}t^b_{gb} \\ -2t^b_{rb}t^b_{gb} & v^b_{rg} \end{pmatrix} = \begin{pmatrix} v^b_{bb} & -2c_0\epsilon^2 \\ -2c_0\epsilon^2 & v^b_{rb} \end{pmatrix}, \qquad (4b)$$

$$\tilde{H}^{(2)} = \begin{pmatrix} v^r_{bb} & -2(t^r_{bb})^2 \\ -2(t^r_{bb})^2 & v^r_{bb} \end{pmatrix} = \begin{pmatrix} v^r_{bb} & -2\epsilon^2 \\ -2\epsilon^2 & v^r_{bb} \end{pmatrix}, \qquad (4c)$$

$$\tilde{H}^{(3)} = \begin{pmatrix} v^g_{bb} & -2(t^g_{bb})^2 \\ -2(t^g_{bb})^2 & v^g_{bb} \end{pmatrix} = \begin{pmatrix} v^g_{bb} & 0 \\ 0 & v^g_{bb} \end{pmatrix}. \qquad (4d)$$

We now tune $\tilde{H}$ to the "small loop" value d by requiring

$$\tilde{H}^{(1)} = \tilde{H}^{(1')} \propto \begin{pmatrix} d & -1 \\ -1 & d^{-1} \end{pmatrix}$$

as these moves change the number of small loops by one [cf. Eq. (3)]. Since a move of type 2 is just an isotopy move, we require $\tilde{H}^{(2)} \propto (\mathbf{I} - \sigma^x)$. Finally, $\tilde{H}^{(3)} = 0$ provided $k > 1$, since it represents a "surgery" on two strands not allowed for $k > 1$. [For $k = 1$, on the other hand, $\tilde{H}^{(3)} \propto (\mathbf{I} - \sigma^x)$ since configurations which differ by such a surgery should have equal coefficients in any GS vector Ψ.] Thus, for $k > 1$ the matrix relations (4a)–(4d) yield equations in the model parameters:

$$\text{types (1)\&(1}') : v^b_{gb} = v^b_{bb} = 2dc_0\epsilon^2 \quad (5a)$$

$$\text{and} \quad v^b_{rb} = v^b_{rg} = 2d^{-1}c_0\epsilon^2, \quad (5b)$$

$$\text{types (2)\&(3)} : v^r_{bb} = 2\epsilon^2 \quad \text{and} \quad v^g_{bb} = 0. \quad (5c)$$

Finally, the bare ring exchange term, T_R, given by Eq. (2) becomes in matrix notation

$$T_R = a\begin{pmatrix} \lambda^2 & -\lambda \\ -\lambda & 1 \end{pmatrix},$$

where $\lambda = d^3$ according to the discussion after Eq. (3). Additionally, the off-diagonal elements of T_R should be of order ϵ^2 in order for this ring exchange to dominate all other ring exchanges that will appear in the higher orders of perturbation theory. (There is, however, some leeway in defining T_R, as discussed in [26].) Along with Eqs. (5), these conditions place our model at the soluble point characterized by d isotopy.

This construction shows how an extended Hubbard model with an additional ring exchange term (or the equivalent quantum dimer model) can be tuned to the d-isotopy state(s). As discussed earlier, they satisfy two of the three conditions that define a class of stable, gapped topological phases which are centered about the special values $d = 2\cos(\pi/k + 2)$. The next step is to understand how perturbations can push the system (by implementing the JW projectors) into these phases. Our simplest candidate for a ''universal quantum computer'' is associated with $d = (1 + \sqrt{5})/2$.

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[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2] S. Das Sarma and A. Pinczek, *Perspectives in Quantum Hall Effects: Novel Quantum Liquids in Low-Dimensional Semiconductor Structures* (Wiley, New York, 1997).
[3] P. W. Anderson, Science **235**, 1196 (1987).
[4] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, Phys. Rev. B **35**, 8865 (1987); D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988).
[5] V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987); R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988); Science **242**, 525 (1988); A. Fetter, C. Hanna, and R. Laughlin, Phys. Rev. B **39**, 9679 (1989); Y. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).
[6] N. Read and B. Chakraborty, Phys. Rev. B **40**, 7133 (1989); N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991); Int. J. Mod. Phys. B **5**, 219 (1991).
[7] X. G. Wen, Phys. Rev. B **44**, 2664 (1991).
[8] C. Mudry and E. Fradkin, Phys. Rev. B **49**, 5200 (1994).
[9] L. Balents, M. P. A. Fisher, and C. Nayak, Int. J. Mod. Phys. B **12**, 1033 (1998); T. Senthil and M. P. A. Fisher, Phys. Rev. B **62**, 7850 (2000).
[10] R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
[11] L. Balents, M. P. A. Fisher, and S. M. Girvin, Phys. Rev. B **65**, 224412 (2002).
[12] T. Senthil and O. Motrunich, Phys. Rev. B **66**, 205104 (2002); O. I. Motrunich and T. Senthil, Phys. Rev. Lett. **89**, 277004 (2002).
[13] R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987); W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **83**, 3530 (1999).
[14] G. Moore and N. Read, Nucl. Phys. **B360**, 362 (1991).
[15] M. Greiter, X. G. Wen, and F. Wilczek, Nucl. Phys. **B374**, 567 (1992).
[16] C. Nayak and F. Wilczek, Nucl. Phys. **B479**, 529 (1996); N. Read and E. Rezayi, Phys. Rev. B **54**, 16 864 (1996); E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, Nucl. Phys. **B516**, 704 (1998).
[17] A. Y. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[18] M. H. Freedman, Found. Comput. Math. **1**, 183 (2001).
[19] L. Kauffman and S. Lins, *Temperley Lieb Recoupling Theory and Invariants of 3-Manifolds*, Annals of Mathematical Studies Vol. 134 (Princeton University Press, Princeton, NJ, 1994).
[20] M. H. Freedman, Commun. Math. Phys. **234**, 129 (2003).
[21] M. Freedman, C. Nayak, and K. Shtengel, cond-mat/0408257.
[22] M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch, and I. Bloch, Nature (London) **415**, 39 (2002).
[23] L. B. Ioffe, M. Feigel'man, V. A. Ioselevich, D. Ivanov, M. Troyer, and G. Blatter, Nature (London) **415**, 503 (2002).
[24] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Ann. Phys. (N.Y.) **310**, 428 (2004).
[25] M. H. Freedman, M. J. Larsen, and Z. Wang, Commun. Math. Phys. **227**, 605 (2002); **228**, 177 (2002).
[26] M. Freedman, C. Nayak, and K. Shtengel, cond-mat/0309120.
[27] M. Roger, Phys. Rev. B **30**, 6432 (1984).
[28] C. Herring, Rev. Mod. Phys. **34**, 631 (1962).
[29] A. H. MacDonald, S. M. Girvin, and D. Yoshioka, Phys. Rev. B **37**, 9753 (1988).
[30] D. J. Thouless, Proc. Phys. Soc. London **86**, 893 (1965).
[31] S. Chakravarty, S. Kivelson, C. Nayak, and K. Voelker, Philos. Mag. B **79**, 859 (1999); K. Voelker and S. Chakravarty, Phys. Rev. B **64**, 235125 (2001).
[32] It is important that the lattice is *not* bipartite. On the edges of a bipartite lattice, our models have an additional, undesired, conserved quantity (integral winding numbers, which are inconsistent with the JW projectors), so the triangular lattice gives the simplest realization.
[33] B. Sutherland, Phys. Rev. B **37**, 3786 (1988).
[34] D. Jetchev (private communication). For a rigorous proof of ergodicity in a similar setting see C. Kenyon and E. Rèmila, Discrete Math. **152**, 191 (1996).