Nondiffusive Transport in Plasma Turbulence: A Fractional Diffusion Approach

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Numerical evidence of nondiffusive transport in three-dimensional, resistive pressure-gradient-driven plasma turbulence is presented. It is shown that the probability density function (pdf) of tracer particles' radial displacements is strongly non-Gaussian and exhibits algebraic decaying tails. To model these results we propose a macroscopic transport model for the pdf based on the use of fractional derivatives in space and time that incorporate in a unified way space-time nonlocality (non-Fickian transport), non-Gaussianity, and nondiffusive scaling. The fractional diffusion model reproduces the shape and space-time scaling of the non-Gaussian pdf of turbulent transport calculations. The model also reproduces the observed superdiffusive scaling.

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Recent experimental and theoretical evidence indicates that transport in magnetically confined fusion plasmas deviates from the standard-diffusion paradigm. Typical examples include the confinement time scaling in low confinement mode plasmas [1,2], perturbative experiments [3–5], and the non-Gaussianity and long-range correlations of fluctuations [6]. The standard-diffusion paradigm breaks down in these cases because it rests on restrictive assumptions including locality, Gaussianity, lack of long-range correlations, and linearity. In particular, according to Fick's law, the fluxes, which contain the dynamical information of the transport process, are assumed to depend only on local quantities, i.e., the gradients of the fields. Also, at a microscopic level, the diffusion paradigm assumes the existence of an underlying uncorrelated, Gaussian stochastic process, i.e., a Brownian random walk. The need to develop models that go beyond these restrictive assumptions is the main motivation of this Letter that has two connected goals. The first goal is to show numerical evidence of nondiffusive transport in pressure-gradient-driven plasma turbulence. We do this by integrating tracer particles in the $\mathbf{E} \times \mathbf{B}$ flow obtained from a nonlinear, three-dimensional turbulence model. Tracer particle studies of this type have the advantage that incorporate in the particle trajectories all the physics of the turbulence model. However, this "microscopic" approach has the limitation of being time consuming, and potentially redundant in the sense that it tracks individual, particle orbit information that from a statistical point of view might be irrelevant. This issue takes us to the second goal which is to propose and test a macroscopic model describing the statistical properties of transport in pressure-gradient-driven plasma turbulence. The proposed model is based on fractional derivative operators which incorporate in a natural, unified way nonlocality in space and time, non-Gaussianity, and anomalous diffusion scaling.

The underlying instability in pressure-gradient-driven plasma turbulence is the resistive interchange mode, driven by the pressure gradient in regions where the magnetic field line curvature is negative. In this system, changes in the pressure-gradient trigger instabilities at rational surfaces that locally flatten the pressure profile and increase the gradient in nearby surfaces. This in turn leads to successive instabilities and intermittent, avalanchelike transport [7], which has been observed to cause anomalous diffusion [8]. This instability is the analog of the Rayleigh-Taylor instability, which is extensively studied in fluids and is responsible for the gravity-driven overturning of a low density fluid laying below a high density fluid. In magnetically confined plasmas the role of gravity is played by the curvature of the magnetic field lines which in cylindrical geometry is always negative and depends only on the radius. The turbulence model that we use describes the coupled evolution of the electrostatic potential Φ and pressure p in a cylinder [7]:

$$\left(\frac{\partial}{\partial \tau} + \tilde{\mathrm{V}} \cdot \nabla + \langle V_\theta \rangle \frac{1}{r} \frac{\partial}{\partial \theta}\right) \nabla_\perp^2 \tilde{\Phi} = -\frac{1}{\eta m_i n_0 R_0} \nabla_\parallel^2 \tilde{\Phi} + \frac{B_0}{m_i n_0} \frac{1}{r_c} \frac{1}{r} \frac{\partial \tilde{p}}{\partial \theta} + \mu \nabla_\perp^4 \tilde{\Phi}, \tag{1}$$

$$\left(\frac{\partial}{\partial \tau} + \tilde{\mathrm{V}} \cdot \nabla + \langle V_\theta \rangle \frac{1}{r} \frac{\partial}{\partial \theta}\right) \tilde{p} = \frac{\partial \langle p \rangle}{\partial r} \frac{1}{r} \frac{\partial \tilde{\Phi}}{\partial \theta} + \chi_\perp \nabla_\perp^2 \tilde{p} + \chi_\parallel \nabla_\parallel^2 \tilde{p}, \tag{2}$$

where the tilde denotes fluctuating quantities (in time and space), and the bracket, $\langle \rangle$, denotes the poloidal and the toroidal angular (flux surface) average. The magnetic field B_0 is assumed to be on a cylinder with the axis along the z axis. The equilibrium density is n_0, the ion mass is m_i, the averaged radius of curvature of the magnetic field lines is r_c, and the resistivity is η. The subindex "$\perp$" denotes the direction perpendicular to the magnetic field, and the subindex "$\|$" denotes the direction parallel to the magnetic field. In both Eqs. (1) and (2) there are dissipative terms with characteristic coefficients μ (the collisional velocity) and $\chi_\perp$ (the collisional cross-field transport). A parallel dissipation term proportional to $\chi_\|$ is also included in the pressure equation. This term can be interpreted as a parallel thermal diffusivity. In all the calculations discussed here, we assume a vanishing poloidal flow velocity, $\langle V_\theta \rangle = 0$. The evolution equation of the flux surface averaged pressure,

$$\frac{\partial \langle p \rangle}{\partial \tau} + \frac{1}{r}\frac{\partial}{\partial r} r\langle \tilde{V}_r \tilde{p} \rangle = S_0 + D \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \langle p \rangle}{\partial r} \right), \quad (3)$$

contains a source term, S_0, which is only a function of r; in the present calculations $S_0 = \tilde{S}_0[1 - (r/a)^2]$. The parameter D represents the collisional transport, and its value is consistent with collisional diffusion. The model parameters used here are $\mu = 0.2a^2/\tau_R$ and $\chi_\perp = 0.025a^2/\tau_R$, where $\tau_R \equiv a^2\mu_0/\eta$ is the resistive time and a the minor radius. The rest of the parameters in the model can be reduced to two dimensionless quantities: the Lundquist number, which is taken to be $S = 10^5$, and $\beta_0/2\varepsilon^2 = 0.018$, where β_0 is the value of β at the magnetic axis and $\varepsilon = a/R_0$. The numerical calculations were carried out with the KITE code [9] using 363 Fourier components to represent the poloidal and toroidal angle dependence for each fluctuating component, and a radial grid resolution of $\Delta r = 7.50 \times 10^{-4} a$.

Having computed the electrostatic potential $\tilde{\Phi}$, we studied transport by following tracer particle orbits determined from the solutions of the equation of motion

$$\frac{d\mathbf{r}}{d\tau} = \tilde{\mathbf{V}} = \frac{1}{B_0^2} \nabla \tilde{\Phi} \times \mathbf{B}_0. \quad (4)$$

Since the magnetic field is fixed, the turbulence-induced transport is only due to the fluctuating electrostatic potential. As an initial condition we used 25 000 tracer particles with random initial positions in θ and z, and radial position $r = 0.5a$. Finite size effects did not seem to be relevant because during the evolution there were very few particles moving out of the domain. It is observed that tracer particles either get trapped in eddies for long times or jump over several sets of eddies in a single flight, giving rise to anomalous diffusion [8]. Our main object of study is the probability density function (pdf) of radial displacements of the particles, $P(x, t)$, where $x = (r - a/2)/a$ and $t = \tau/\tau_R$. As t evolves, the pdf broadens and develops tails. The triangles in Fig. 1 show $P(x, t)$ at $t = 0.64$ in a log-normal scale, obtained from the histogram of particle displacements. The numerical results show that for times above $t = 0.1$ the moments of the tracer particle displacements exhibit superdiffusive scaling, $\langle x^n \rangle \sim t^{n\nu}$, with $\nu = 0.66 \pm 0.02$.

The generic form of the proposed model is

$${}^c_0D_t^\beta P = \chi[w^- \, {}_aD_x^\alpha + w^+ \, {}_xD_b^\alpha]P + \Lambda, \quad (5)$$

where Λ is a source,

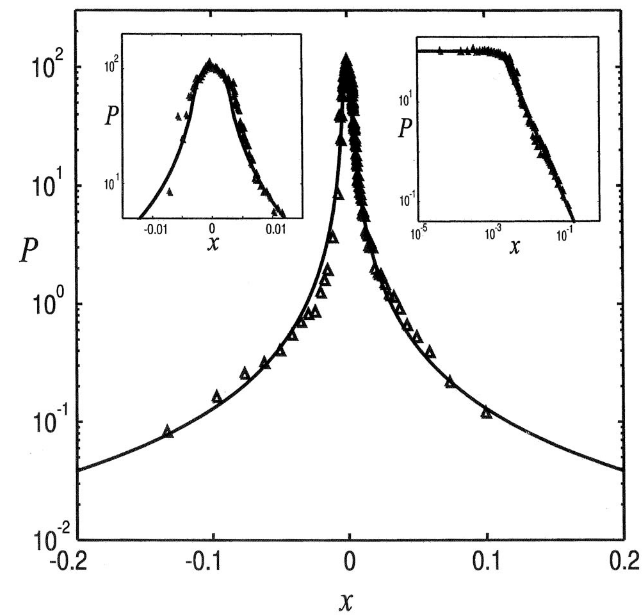

FIG. 1. Non-Gaussian probability density function of tracer particles in plasma turbulence. The triangles denote the results from the turbulence model in Eqs. (1)–(4). The solid line is the analytical solution in Eq. (11) of the symmetric ($w^+ = w^-$) fractional diffusion model in Eq. (5) with $\alpha = 3/4$, $\beta = 1/2$, and $\chi = 0.09$. The inset on the left shows the parabolic dependence of the core of the pdf according to Eq. (13), and the log-log inset on the right shows the algebraic decay of the tails with exponent $1 + \alpha = 7/4$.

$${}_aD_x^\alpha P = \frac{1}{\Gamma(m - \alpha)} \partial_x^m \int_a^x \frac{P(y, t)}{(x - y)^{\alpha + 1 - m}} dy, \quad (6)$$

$${}_xD_b^\alpha P = \frac{(-1)^m}{\Gamma(m - \alpha)} \partial_x^m \int_x^b \frac{P(y, t)}{(y - x)^{\alpha + 1 - m}} dy, \quad (7)$$

are the left and right Riemann-Liouville fractional derivatives, respectively, $w^\pm$ are weighting factors, and $m - 1 \leq \alpha < m$ with m a positive integer. The operator on the left-hand side of Eq. (5) is the Caputo fractional derivative in time of order $0 < \beta < 1$,

$${}^c_0D_t^\beta P = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{\partial_\tau P(x, \tau)}{(t - \tau)^\beta} d\tau. \quad (8)$$

Fractional derivatives are natural generalizations of regular derivatives. As expected, for α and β integers, these operators reduce to regular derivatives, and results of regular calculus extend directly to the fractional domain [10]. The fractional model can be equivalently written in the flux conservative from $\partial_t P = -\partial_x[w^-\Gamma_\ell + w^+\Gamma_r]$, where $\Gamma_\ell = -\chi \, {}_aD_x^{\alpha-1} \, {}_0D_t^{1-\beta} P$ and $\Gamma_r = \chi \, {}_xD_b^{\alpha-1} \, {}_0D_t^{1-\beta} P$. According to this, non-Fickian effects due to avalanchelike events that induce large displacements of tracers are described using nonlocal, integro-differential operators in space. The flux at x consists of a "left-sided" contribution, Γ_ℓ, from the (a, x) interval, and a "right-sided" contribution, Γ_r, from the (x, b) interval. The time integrals in the fluxes account for non-Markovian, "memory" effects due to the trapping of tracers in eddies. The parameters $w^{\pm}$ determine the relative weight of the fluxes. Because the individual tracers follow the turbulent velocity field $\tilde{\mathbf{V}}$ according to Eq. (4), the pdf of the tracers satisfies the passive scalar equation $\partial_t P = -(\tilde{\mathbf{V}} \cdot \nabla)P$. In this regard, the results presented here represent a first step in a phenomenological renormalization of $\tilde{\mathbf{V}} = \mathbf{E} \times \mathbf{B}$ turbulent transport using fractional transport operators according to the prescription $\tilde{\mathbf{V}} \cdot \nabla \rightarrow \partial_x[w^- \Gamma_\ell + w^+ \Gamma_r]$.

The physics behind the model in Eq. (5) can be further understood from the close connection with the theory of random walks. The standard-diffusion model is a macroscopic description of the Brownian random walk which assumes that at fixed time intervals $t = T, 2T, \ldots, nT, \ldots$ particles at a microscopic level experience an uncorrelated random displacement, or jump, ℓ_n, with probability $\mathcal{P}_x$, where $\mathcal{P}_x$ is assumed to have a finite second moment. In a similar way, fractional diffusion models can be viewed as macroscopic descriptions of generalized Brownian random walks known as continuous time random walk (CTRW) models [11]. In addition to the jump probability density $\mathcal{P}_x$, the CTRW model introduces a waiting time probability function $\mathcal{P}_t$. The different types of CTRW processes, and the resulting macroscopic transport models, can be classified based on the characteristic waiting time, T, and the characteristic mean-square jump, σ^2, being finite or divergent [11]. Based on this, the model (5) can be understood as a general macroscopic description of an underlying microscopic stochastic process in which particles exhibit both jumps without a characteristic spatial scale and waiting times without a characteristic time scale. The space nonlocality is a direct consequence of the existence of anomalously large jumps (known also as Levy flights) that connect distant regions in space, and the time nonlocality is due to the history dependence introduced in the dynamics by the presence of anomalously large waiting times.

The fractional diffusion model in Eq. (5) is fairly general, and, depending on the values of α, β, and $w^{\pm}$, different transport processes can be modeled, including subdiffusive transport, superdiffusive transport, and asymmetric transport. In what follows we show that, for the symmetric, superdiffusive transport observed in pressure-gradient-driven turbulence, $w^+ = w^- = 1/\sqrt{2}$, $\alpha = 3/4$, $\beta = 1/2$, and $\Lambda = 0$. In this case, the solution of Eq. (5) in the infinite domain $x \in (-\infty, \infty)$ can be written as

$$P(x, t) = \int_{-\infty}^{\infty} \mathcal{G}(x - x', t) P(x', 0) dx', \tag{9}$$

where $\mathcal{G}(x, t) = (\chi^{1/\beta} t)^{-\beta/\alpha} K(\eta)$ is the Green's function of the fractional diffusion equation,

$$K(\eta) = \frac{1}{\pi} \int_0^{\infty} \cos(\eta z) E_\beta(-z^\alpha) dz, \tag{10}$$

E_β is the Mittag-Leffler function, and $\eta = x(\chi^{1/\beta} t)^{-\beta/\alpha}$ is a similarity variable [10,12]. As expected, for $\alpha = 2$ and $\beta = 1$, $\mathcal{G}$ reduces to a Gaussian. For $\beta = 1$, $1 < \alpha \leq 2$, $\mathcal{G}$ becomes a symmetric Levy stable distribution [13], and, for $0 < \beta < 1$, $\alpha = 2$, it reduces to the solution of the subdiffusion fractional equation [11]. Consistent with the initial condition used in the tracer particle calculation, we consider $P(x, 0) = P_0/\varepsilon$, for $|x| \leq \varepsilon/2$ and 0 otherwise, with $\varepsilon \ll 1$, and write the solution as

$$P(x, t) = \frac{1}{2\delta} (\chi^{1/\beta} t)^{-\beta/\alpha} \int_{\eta - \delta}^{\eta + \delta} K(z) dz, \tag{11}$$

where $\delta = (\varepsilon/2)(\chi^{1/\beta} t)^{-\beta/\alpha} \ll 1$. Outside the support of the initial condition, $|x| > \varepsilon/2$, Eq. (11) can be written as a power series in the small parameter δ:

$$P(x, t) = (\chi^{1/\beta} t)^{-\beta/\alpha} \left[K(\eta) + \frac{\delta^2}{3!} K''(\eta) + \frac{\delta^4}{5!} K''''(\eta) + \ldots \right]. \tag{12}$$

From Eq. (12), using the asymptotic properties of K, it follows that, for $x \gg (\chi^{1/\beta} t)^{\beta/\alpha}$, $P(x, t) \sim \chi t^\beta x^{-(1+\alpha)}[1 + (\varepsilon^2/4!)(\alpha + 1)(\alpha + 2)x^{-2} \ldots]$. That is, the tracers' pdf exhibits algebraic tails with the decay exponent equal to $1 + \alpha$. On the other hand, in the $|x| < \varepsilon/2$ region, Eq. (11) gives, in the small δ limit,

$$P(x, t) = \frac{A}{\alpha \chi t^\beta} \left(\frac{2}{\varepsilon} \right)^{1-\alpha} \left[1 - 2\alpha(1 - \alpha) \left(\frac{x}{\varepsilon} \right)^2 \right] + \frac{B}{(\chi^{1/\beta} t)^{\beta/\alpha}}, \tag{13}$$

where A and B are known constants determined from the small η limit of the function K. As expected, near the origin, the core of the distribution exhibits a parabolic profile. Figure 1 shows the quantitative agreement between the numerical data from the tracer particles in the 3D turbulence calculations, and the solution of the fractional diffusion model according to Eqs. (12) and (13). The index β determines the time-asymptotic scaling properties of P. To show this, we introduce the time-scaling variable $\zeta = \chi^{1/\beta} t |x|^{-\alpha/\beta}$, and write the solution as

$$P = |x|^{-1} \zeta^{-\beta/\alpha} K(\zeta^{-\beta/\alpha}). \tag{14}$$

Using again the large η, and also the small η asymptotic behavior of the function $K(\eta)$, it follows that $P(x_0, t) \sim t^\beta$, for $\chi^{1/\beta} t \ll |x_0|^{\alpha/\beta}$, and $P(x_0, t) \sim t^{-\beta}$, for $\chi^{1/\beta} t \gg |x_0|^{\alpha/\beta}$. This scaling is verified in Fig. 2 that shows the evolution in time of P at a fixed position x_0. The analytical solution according to Eq. (14), shown with a solid line, exhibits algebraic tails in the small t and large t limits, and the expected peak at intermediate times. The circles and the triangles in the figure denote the results obtained from the tracer particle turbulence simulations. The agreement is good, but not as sharp as the one in Fig. 1 due to the

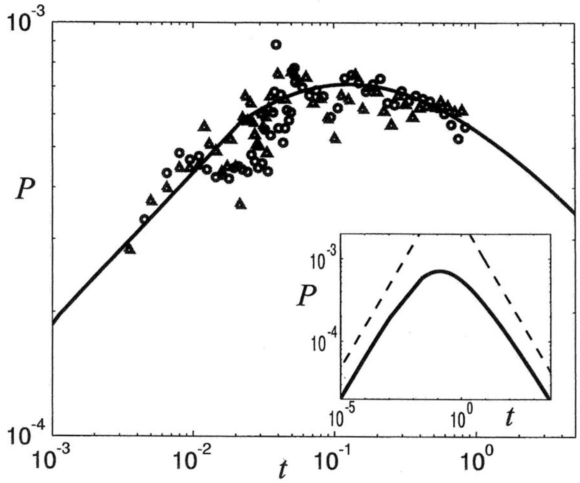

FIG. 2. Time evolution of the probability density function of tracer particles in plasma turbulence. The circles and the triangles denote the results from the turbulence model in Eqs. (1)–(4). The solid line is the analytical solution in Eq. (14) of the symmetric ($w^+ = w^-$) fractional diffusion transport model in Eq. (5) with $\alpha = 3/4$, $\beta = 1/2$, and $\chi = 0.09$. In agreement with the asymptotic result, the dashed lines in the inset show an algebraic dependence of the tails with exponent $|\beta| = 1/2$.

numerical limitations in the integration of the turbulence model for large times. The moments of the tracer particle displacements scale as $\langle x^n \rangle \sim t^{n\nu}$, with $\nu = \beta/\alpha$. For $\alpha = 3/4$and $\beta = 1/2$, $\nu = 2/3$, a value in very good agreement with the one obtained in the turbulence simulation, $\nu = 0.66 \pm 0.02$. The superdiffusive scaling implies an anomalous confinement time scaling $t_c \sim a^{3/2}$, a reasonable value in the range of the experimentally determined values which typically deviate from the standard-diffusion prediction $t_c \sim a^2$ [2].

To summarize, we have presented numerical evidence of nondiffusive transport of tracers in three-dimensional, resistive, pressure-gradient-driven plasma turbulence. The pdf of displacements is strongly non-Gaussian, with algebraic tails, and the moments exhibit superdiffusive scaling. We proposed a macroscopic transport model with fractional derivative operators of order $\alpha = 3/4$ in space, and order $\beta = 1/2$ in time. The model incorporates, in a unified way, space nonlocality (non-Fickian transport), memory effects (non-Markovian transport), and anomalous diffusion scaling. There is quantitative agreement between the model and the turbulence transport calculations. The plasma turbulence model used in the calculations is the same as the one used in Ref. [7] to explore the role of SOC (self-organized criticality) in plasma confinement. In this regard, the present results provide evidence that some aspect of SOC-like systems can be described with fractional transport models. We have restricted attention to passive scalars. As a first step to study the transport of active scalars, we have followed the evolution of localized pressure pulses in nonsteady turbulence. Preliminary results indicate that positive and negative pressure pulses exhibit superdiffusive scaling with the same anomalous diffusion exponent found in the passive scalar problem [14]. However, the active nature of the pressure manifests in interesting asymmetries in the spreading of the pulses that remain to be described within the framework of fractional models. Most likely, this will require the use of asymmetric, $w^- \neq w^+$, fractional operators that incorporate pinch effects and asymmetric tails. An interesting and important problem is to understand the interplay of nonlinearity and fractional diffusion. As a first step in this direction we have added to Eq. (5) a nonlinearity of the form $P(1 - P)$, typically used in reduced models of the $L - H$ transition (e.g., Ref. [15]). Numerical and analytical results on this model indicate that nonlinearity, and the nonlocality of fractional diffusion, lead to an exponential propagation of fronts [16]. These results might be relevant in the study of rapid propagation phenomena in the $L - H$ transition.

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