Nonlinear Evolution of $q = 1$ Triple Tearing Modes in a Tokamak Plasma

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In magnetic configurations with two or three $q = 1$ (with q being the safety factor) resonant surfaces in a tokamak plasma, resistive magnetohydrodynamic modes with poloidal mode numbers m much larger than 1 are found to be linearly unstable. It is found that these high-m double or triple tearing modes significantly enhance through nonlinear interactions the growth of the $m = 1$ mode. This may account for the sudden onset of the internal resistive kink, i.e., the fast sawtooth trigger. Based on the subsequent reconnection dynamics that can proceed without formation of the $m = 1$ islands, it is proposed that high-m triple tearing modes are a possible mechanism for precursor-free partial collapses during sawtooth oscillations.

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The profile of the safety factor $q(r)$ (which measures the magnetic field line pitch) contains information about the instability characteristics of magnetically confined plasmas in toroidal or helical systems with respect to current-driven magnetohydrodynamic (MHD) instabilities. In particular, in the presence of magnetic surfaces where $q = 1$, a mixing of the plasma inside these surfaces was observed. This instability is thought to be closely related to internal disruptions, generally known as sawtooth oscillations, which strongly affect the quality of energy and particle confinement. In view of the desired application to thermonuclear fusion reactors such as ITER, a detailed understanding of these internal large-scale instabilities is necessary.

A heuristic model proposed by Kadomtsev [1] successfully explains overall phenomena associated with a full sawtooth crash. In this model, a perturbation with helicity $h = m/n = 1$ (m being the poloidal and n the toroidal Fourier mode number), which is in resonance with the closed field lines on the $q = 1$ surface, quenches the hot core region inside the $q = 1$ surface through magnetic reconnection. This kind of relaxation, generally known as the $m = 1$ internal resistive kink instability (in short, the $m = 1$ mode), has been observed to exhibit an abrupt onset, which is called a "fast trigger." However, a satisfactory explanation of this fast trigger is not yet known. Another unresolved problem is the possibility of a partial sawtooth collapse, where the $m = 1$ mode saturates before the reconnection of the core is completed, so only an annular (off-axis) region undergoes mixing [2].

During the evolution of a tokamak plasma subject to sawtooth relaxation oscillations, multiple $q = 1$ resonant surfaces may arise temporarily [3]. When this occurs in configurations with a hollow current profile ($q_0 > 1$ with q_0 being the q at the magnetic axis), $q = 1$ double tearing modes (DTMs) can become unstable [4–6] whereas, for a centrally peaked current profile ($q_0 < 1$), $q = 1$ triple tearing modes (TTMs) may arise [6].

In this Letter, it is demonstrated that some of the phenomena associated with the sawtooth oscillations in tokamaks can be explained by the nonlinear evolution of DTMs or TTMs. Using nonlinear numerical simulations, it is shown that, in the presence of multiple $q = 1$ resonant surfaces, rapidly growing high-m DTMs or TTMs can enhance the growth of the $m = 1$ mode and later generate electromagnetic turbulence in the annular region surrounded by the $q = 1$ resonant surfaces. Based on these observations, it is shown that the fast trigger of a sawtooth crash as well as precursor-free partial collapses during sawtooth relaxation oscillations can be accounted for by the nonlinear evolution of TTMs.

The set of equations we use is the reduced magnetohydrodynamic (RMHD) equation in the zero-beta limit in a cylindrical geometry. In normalized form the RMHD model can be written as

$$\partial_t \psi = [\psi, \phi] - \partial_\zeta \phi - S_{\mathrm{Hp}}^{-1}(\hat{\eta} j - E_0), \tag{1}$$

$$\partial_t u = [u, \phi] + [j, \psi] + \partial_\zeta j + \nu \nabla_\perp^2 u, \tag{2}$$

where ψ is the magnetic flux function, ϕ the stream function (electrostatic potential), $j = -\nabla_\perp^2 \psi$ the axial current density, and $u = \nabla_\perp^2 \phi$ the vorticity, essentially following the standard notation (cf., e.g., Ref. [7]). The time t is normalized by the poloidal Alfvén time τ_{Hp} (time scale for dynamics in an ideal magnetized plasma) and the radial coordinate by the minor radius a of the plasma column. The resistivity profile is given by $\hat{\eta}(r) = j(r = 0, t = 0)/j(r, t = 0)$. As to the Lundquist number $S_{\mathrm{Hp}} = \tau_{\mathrm{R}}/\tau_{\mathrm{Hp}}$ [where $\tau_{\mathrm{R}} = \mu_0/(\eta_0 a^2)$ is the resistive time scale and η_0 the resistivity at $r = 0$] and viscosity ν (normalized by a^2/τ_{Hp}), $S_{\mathrm{Hp}} = \nu^{-1} = 10^6$ is used, unless stated otherwise. The constant source term E_0 compensates the resistive dissipation of the equilibrium current. Using a quasispectral code with a finite-difference radial mesh, single helicity ($h = m/n = 1, 0 \leq m \leq 127$) nonlinear simulations of

Eqs. (1) and (2) were performed with an initial condition corresponding to a flowless equilibrium with perturbed flux function [$\psi_{m>0}(t=0) = 10^{-11}$, $m = n$]. An ideally conducting wall was taken as the boundary condition. Although the RMHD model encompasses only part of the physical effects involved in sawtooth oscillations (cf., e.g., Refs. [2,8,9]), it suffices for the present purpose of following the fundamental dynamics of DTMs and TTMs.

The important new feature of *linear* instability that is addressed here is the fact that configurations with multiple $q = 1$ resonant surfaces in general possess a *broad spectrum* of linearly unstable modes. Moreover, the fastest growing mode often has a poloidal mode number $m \sim \mathcal{O}(10)$. To show this, the q profile shown in Fig. 1 is employed, where three $q = 1$ resonant surfaces are present, at the radii $r_{s1} < r_{s2} < r_{s3}$. By evolving the linearized RMHD equations in time, spectra of linear growth rates (i.e., dispersion relations) $\gamma_{\mathrm{lin}}(m)$ were obtained as functions of m, as plotted in Fig. 2 for Lundquist numbers $S_{\mathrm{Hp}} = 10^6$, 10^7 and 10^9. Hereby the Prandtl number Pr $= S_{\mathrm{Hp}}\nu$ has been kept equal to unity. Clearly, a variation of S_{Hp} (while Pr $= 1$) retains the broadness of the spectrum and $\gamma_{\mathrm{max}} = \max\{\gamma_{\mathrm{lin}}(m)\}$ is located at $m > 1$ in all cases.

The dependence of the spectrum $\gamma_{\mathrm{lin}}(m)$ on the distance between the $q = 1$ surfaces is most easily investigated by considering DTM configurations where two $q = 1$ resonant surfaces are present, located at radii r_{s1} and r_{s2}, a distance $D_{12} = r_{s2} - r_{s1}$ apart. In Fig. 3 the DTM growth rate spectra for profiles with $D_{12} = 0.05$, 0.1, and 0.2 are shown. While varying D_{12}, the local magnetic shears at the resonant radii, $s_1 = s(r_{s1})$ and $s_2 = s(r_{s2})$, were not changed. It can be seen that the narrower the interresonance region becomes, the more the poloidal mode number of the fastest growing mode shifts to larger values of m, and γ_{max} increases. On the other hand, the growth rate of the $m = 1$ mode hardly depends on D_{12} and it becomes the

fastest growing mode for sufficiently large values of D_{12}. Similar results are found for TTMs, whereby it is noted that TTMs tend to peak at higher m with higher growth rates than DTMs with similar D_{ij}. The spectra shown in Figs. 2 and 3 contrast with that for the single tearing modes (STM), for which the $m = 1$ mode is dominant and the modes with higher m are usually linearly stable. Note that the broad spectra of tokamak TTMs that were obtained here are similar to those of tearing modes obtained by Dahlburg and Karpen [10] for triple current sheets in slab geometry as a model for adjoining helmet streamers in the solar corona.

It must be emphasized that the dispersion curves in Fig. 2 show only the growth rate of the most unstable mode for each m. However, in general, for a given m there are up to three unstable TTM eigenmodes, each associated with a resonant surface. To illustrate this, the radial structure of the eigenmodes for $m = 1$ and $m = 13$ are shown in Fig. 4, which were obtained by solving the eigenvalue problem for the linearized equations (1) and (2). Here $M_1(m)$ denotes the eigenmode with the poloidal mode number m that extends only to the innermost resonant surface r_{s1}. Note that $M_1(1)$ has the same mode structure as an STM. Similarly, $M_2(m)$ and $M_3(m)$ denote the eigenmodes that extend to r_{s2} and r_{s3}, respectively. It is also noted that, for $m = 13$, M_1 (not shown) is stable, as can be expected from the linear stability of STMs with higher m. Similar eigenmode structures are found for DTMs, and indeed similar instability characteristics are also expected for q profiles with more than three $q = 1$ resonant surfaces.

After perturbing a large number of unstable modes at random poloidal angles, the $m = 1$ mode evolves as shown in Fig. 5. The most remarkable feature here is the presence of a phase of *nonlinearly driven growth*. There, the energy of the $m = 1$ perturbed mode grows exponentially as
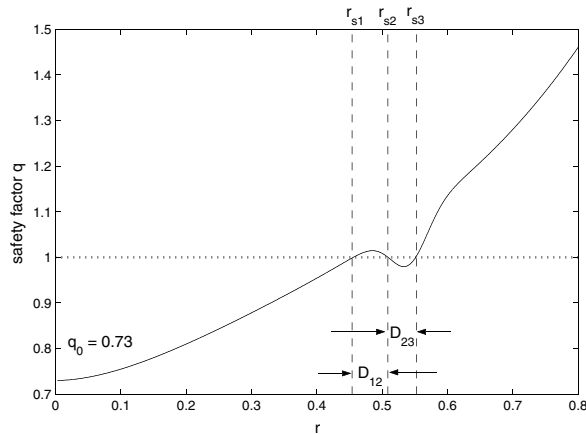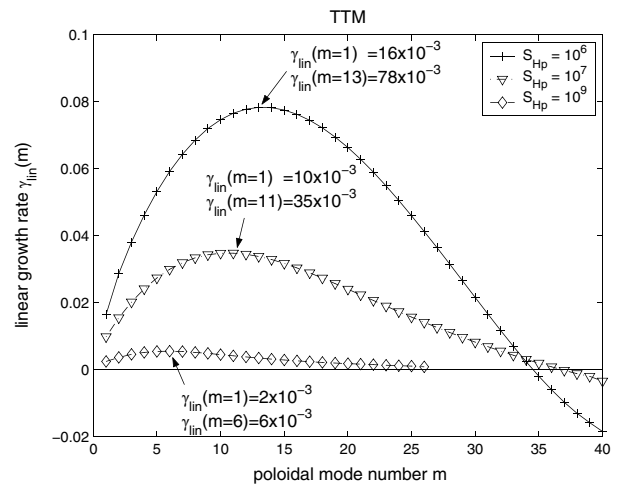

FIG. 1.   A safety factor profile $q(r)$ unstable to TTMs. Vertical lines (dashed) indicate the locations of the $q = 1$ resonant surfaces, $r_{s1} < r_{s2} < r_{s3}$, and $q = 1$ is indicated by a horizontal (dotted) line.

FIG. 2.   Linear growth rate spectra of TTMs for the q profile given in Fig. 1 for the Lundquist numbers $S_{\mathrm{Hp}} = 10^6$, 10^7, and 10^9 and constant Prandtl number Pr $= S_{\mathrm{Hp}}\nu = 1$.
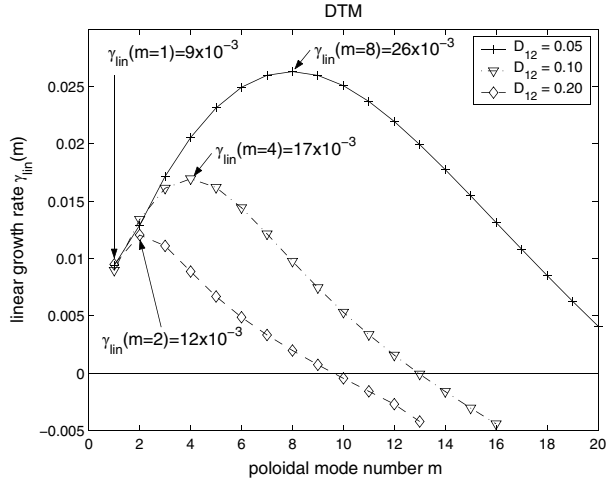
FIG. 3.   Linear growth rate spectra of DTMs for q profiles with two $q = 1$ resonant surfaces located a distance D_{12} apart, obtained with $S_{Hp} = \nu^{-1} = 10^6$. In the three cases shown, $D_{12} = 0.05$, 0.1, and 0.2, respectively.

$\exp(\gamma_{drive}t)$. In this example, $\gamma_{drive} \approx 0.16$, i.e., the $m = 1$ mode grows at a rate which is one order of magnitude larger than its linear growth rate $\gamma_{lin}(m = 1) = 16 \times 10^{-3}$. The nonlinear growth rate γ_{drive} approximately equals twice the maximum growth rate in the spectrum (Fig. 2), $\gamma_{max} = 0.08$, because it results from the nonlinear coupling of m and $m + 1$ mode pairs.

The growth rates shown in Fig. 2 belong exclusively to $M_3(m)$ modes, i.e., the modes extending to the outermost resonant surface at $r = r_{s3}$, since it was assumed here that $s_1 < s_3$ ($s_1 = 0.35$, $s_2 = -0.56$, $s_3 = 1.20$). On the other hand, for a q profile with $s_1 > s_3$, the eigenmode $M_1(m = 1)$ has a higher growth rate than $M_3(m = 1)$. However, also in this case, the highest growth rate among all m modes,

γ_{max}, is typically several times larger than $\gamma_{lin}(m = 1)$ and therefore the nonlinearly driven growth of the $m = 1$ mode still exceeds its linear growth. It is concluded that the rapid nonlinear growth of TTMs occurs in a wide range of TTM q profiles. Since the sawtooth crash is generally considered to be triggered by the onset of an $m = 1$ mode [9], the results presented here suggest that the nonlinear growth of TTMs (or DTMs for a hollow current profile) is one of the possible mechanisms for experimentally observed abrupt sawtooth crashes, i.e., the fast trigger. Let us note that coupled tearing modes on two nearby resonant surfaces can exhibit exponential non-Rutherford growth to saturation even for $q > 1$ [4,11].

Finally, the fully nonlinear regime of Fig. 5 is discussed. At about $t = 200$ electromagnetic turbulence starts to develop in the whole interresonance region $r_{s1} < r < r_{s3}$, as can be seen in Fig. 6(a). The fluctuations flatten the q profile through simultaneous magnetic reconnection in the *whole* interresonance region. This *partial collapse* (i.e., not involving the core) is essentially completed around $t = 220$–230. A contour plot showing the situation after the q profile was annularly flattened is given in Fig. 6(b). An important property of an annular collapse as the one shown in Fig. 6 is that no $m = 1$ islands needs to form and therefore the displacement of the core plasma can be rather small.

This phenomenon is similar to the experimentally observed off-axis temperature collapses without precursor oscillations. For example, Edwards *et al.* [12] reported JET experiments where rapid sawtooth crashes without evident precursor oscillations, but preceded by a partial crash in an off-axis region, were observed.

A possible explanation for such phenomena was proposed by Buratti *et al.* in terms of a "purely growing
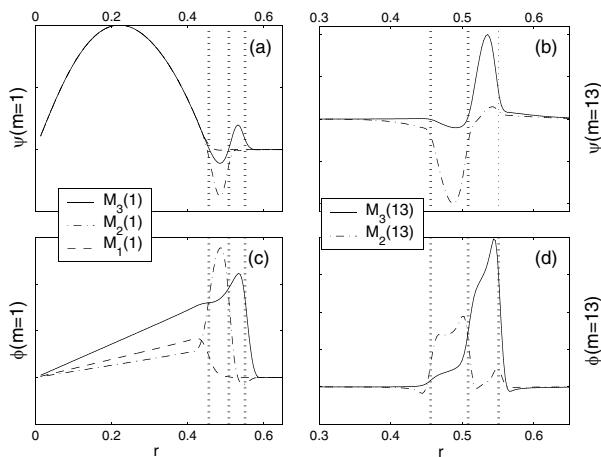

FIG. 4.   Radial structure of linearly unstable TTM eigenmodes for (a) the $m = 1$ modes of ψ, (b) the $m = 13$ modes of ψ, (c) the $m = 1$ modes of ϕ, and (d) the $m = 13$ modes of ϕ obtained for $S_{Hp} = \nu^{-1} = 10^6$. For a given m, the eigenmode extending to the resonant radius r_{si} is denoted by $M_i(m)$. Vertical lines (dotted) indicate resonant radii.
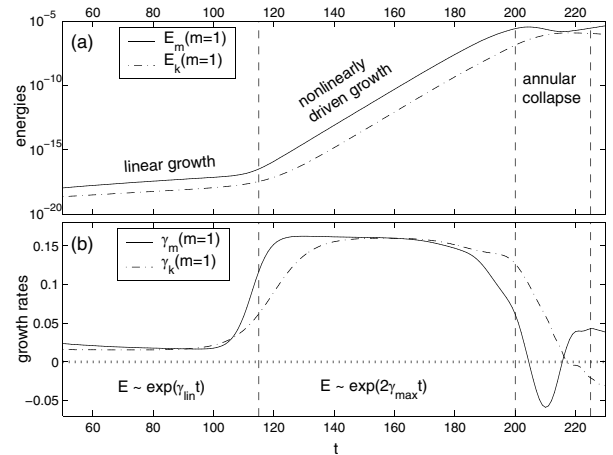

FIG. 5.   Evolution of the $m = 1$ TTM for the q profile given in Fig. 1. The perturbed kinetic and magnetic energies, denoted by E_k and E_m, are plotted in (a). The growth rates γ_k and γ_m given in (b) are estimated from E_k and E_m, and $\gamma_{max} \equiv \max\{\gamma_{lin}(m)\}$ is the maximum growth rate in the spectrum in Fig. 2 for $S_{Hp} = 10^6$. The system reaches the fully nonlinear saturation in the phase denoted as "annular collapse," i.e., $t > 200$.
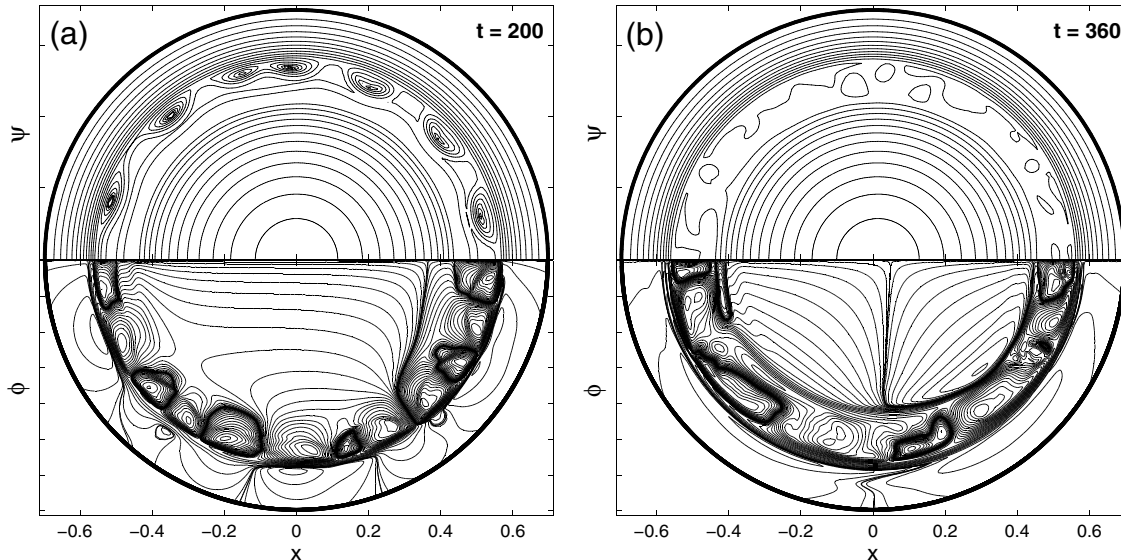
FIG. 6.   Upper and lower halves of ψ (top) and ϕ (bottom) contour lines in the poloidal cross section during the annular collapse (a) and after the q profile was annularly flattened (b).

precursor'' [13]. However, the results in Fig. 6 show that, if a q profile with relatively small interresonance distances D_{12} and D_{23} is formed even temporarily, rapidly growing nonlinear TTMs can also cause such partial crashes without precursors.

Precursor-free partial collapses during the ramp phase of compound sawtooth oscillations were also observed in tokamak discharges with hollow q profiles [14–17], where $q = 1$ DTMs can be expected. In analogy with the TTM case discussed above, our results indicate that nonlinear growth of $q = 1$ DTMs can lead to such a partial collapse. In cases where the growth rate spectrum peaks at lower m (e.g., due to larger D_{ij} or higher S_{Hp}), we have also observed partial collapses (i.e., collapses in the interresonance region) after which the core displacement continues to grow, which is similar to experimental observations made in JET, given in Figs. 4(A)–4(C) in Ref. [12].

In summary, our numerical simulations have shown that the simultaneous excitation of unstable $q = 1$ TTMs with high m and their subsequent nonlinear interactions lead to a rapid onset of the $m = 1$ triple tearing mode, which qualitatively depicts the fast triggering of sawtooth crashes observed in tokamak experiments. Similar behavior has also been found in simulations of $q = 1$ DTMs. If more than three $q = 1$ resonant surfaces are formed in a tokamak discharge, we also expect similar multiple tearing modes that grow rapidly with high poloidal mode numbers. We have also presented a scenario, where the nonlinear evolution of many unstable TTMs leads to a partial collapse of a sawtooth without being preceded by an $m = 1$ precursor. Similar phenomena were observed during compound sawtooth oscillations in several experiments [12,14–17].

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