Full Current Statistics in the Regime of Weak Coulomb Interaction

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We evaluate the full statistics of the current via a Coulomb island that is strongly coupled to the leads. This strong coupling weakens Coulomb interaction. We show that in this case the effects of the interaction can be incorporated into the renormalization of transmission eigenvalues of the scatterers that connect the island and the leads. We evaluate the Coulomb blockade gap in the current-voltage characteristics, the value of the gap being exponentially suppressed as compared to the classical charging energy of the island.

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There are two important manifestations of the quantization of the electric charge. The first is a current shot noise [1]. In mesoscopic conductors with large conductance $G \gg G_Q$, $G_Q = e^2/2\pi\hbar$ being the conductance quantum, the quantum mechanical Pauli principle modifies the shot noise with respect to its classical Poissonian value $S = 2eI$. It also changes the full counting statistics (FCS) of charge transfer, so that this statistics is not the simple Poissonian one [2,3]. The second manifestation of electron charge quantization is the Coulomb blockade. It is most strong provided $G \ll G_Q$ [4]. The FCS in this strongly interacting case is that of a *classical* stochastic Markov process [5]. In this Letter, we address the opposite limit of weak Coulomb interaction, $G \gg G_Q$, where *quantum mechanics* is important.

It has been understood that the charge quantization survives even in the limit $G \gg G_Q$ [6–8]. The free energy of the Coulomb island was shown to retain the periodic dependence on the induced "offset" charge q, thus indicating the Coulomb blockade. However, quantum fluctuations of charge result in exponential suppression of the effective charging energy $\tilde{E}_C$ as compared to its classical value $E_C = e^2/2C$. Most research in this field has been concentrated on the system with tunnel contacts. Albeit, the weak charge quantization persists for arbitrary mesoscopic scatterers, connecting the island and the leads [8]. It is this general and most interesting situation that we address in this Letter. The quantization completely vanishes only for constrictions with perfectly transmitting channels [7]. References [6–8] reflect the milestones in the understanding of this remarkable point. The weak Coulomb blockade in single-electron transistor (SET) with tunnel contacts has been observed in experiments [9,10]. The SET with contacts made of diffusive microstripes has been reported in Ref. [11].

Recent studies link the shot noise in the conductor to the negative interaction correction to the conductance [12] stressing the common nature of both phenomena. The interaction correction to FCS was analyzed recently in [13] for a scatterer embedded in the electromagnetic environment. The relation between interaction correction to the conductance and the formation of Coulomb blockade in an island was addressed in [14].

In this Letter, we evaluate the FCS via a Coulomb island defined by several arbitrary mesoscopic scatterers under conditions of weak Coulomb blockade, $G \gg G_Q$. Our results can be summarized as follows. At energy scale $E \ll g_0 E_c$, $g_0 = G_0/G_Q$ being the dimensionless conductance of the system in the absence of interaction; the dominant effect of Coulomb interaction is the energy-dependent renormalization of the transmission eigenvalues $T_n^{[k]}$ of the mesoscopic scatterers labeled by k,

$$\frac{dT_n^{[k]}}{d\ln E} = \frac{2T_n^{[k]}(1 - T_n^{[k]})}{\sum_{n,k} T_n^{[k]}}. \quad (1)$$

The renormalization of a similar form was previously obtained in [15] for a scatterer in the weakly interacting 1D gas and in [13] for a single multichannel scatterer shunted by an external impedance. We thus prove this simple relation for a Coulomb island. The FCS is readily obtained from the energy-dependent $T_n^{[k]}$ with using noninteracting scattering theory approach of [2,16]. This gives the voltage dependence of conductance, shot noise, and all

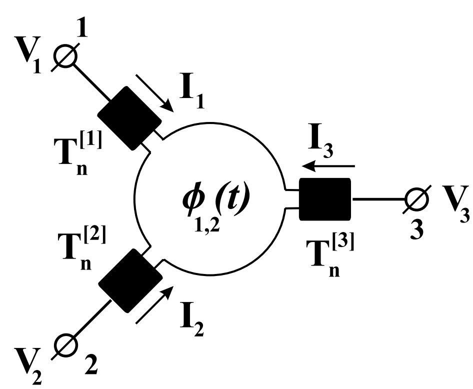

FIG. 1. Multiterminal ($M = 3$) Coulomb island, defined by M multichannel scatterers with transmission coefficients $T_n^{[k]}$.

higher cumulants of charge transfer. In contrast to the case of a single scatterer, the renormalization may break down at finite energy $\tilde{E}_C \propto g_0 E_C e^{-\alpha g_0}$, α being a numerical factor depending on the details of the initial transmission distribution. Remarkably, $\tilde{E}_C$ coincides with the equilibrium effective charging energy evaluated with instanton technique [8]. However, the renormalization stops at the effective Thouless energy $E_{\rm Th} \sim G(E)\delta/G_Q$, δ being mean level spacing in the island. This gives rise to *two* distinct scenarios at low energy. If $g_0 > \alpha^{-1}\ln(E_C/\delta)$, Coulomb blockade does not occur with zero-bias conductance being saturated at the value $G(E_{\rm Th}) \gg G_Q$. Alternatively, $G(0) \approx G_Q$ and $\tilde{E}_C$ defines the Coulomb gap.

Model and the effective action.—We consider the Coulomb island with a charging energy E_C and a mean level spacing $\delta, E_C \gg \delta$. It is connected to $M \geq 2$ external leads by means of M arbitrary mesoscopic scatterers (Fig. 1) that are defined by the set of transmission eigenvalues $T_{n,0}^{[i]}$. We assume that $g_0 = \sum_{n,i} T_{n,0}^{[i]} \gg 1$. Our goal is to evaluate the cumulant generating function (CGF) $S([\chi_i])$, which depends on the set of auxiliary variables $\{\chi_i\}$. The Fourier transform of $\exp(-S)$ with respect to χ_i gives the probability $P(\{N_i\})$ for $N_i \gg 1$ electrons to be transferred to the terminal i during time interval t_0. (See [2].) The derivatives of S give the average currents, shot-noise correlations, and higher order moments of charge transferred.

We evaluate the CGF by extending the semiclassical approach for the FCS of the noninteracting electrons [16]. To account for Coulomb interactions, we introduce a dynamical phase variable $\varphi(t)$ [17] that results from the Hubbard-Stratonovich transform of the charging energy. Its time derivative, $\dot{\varphi}(t)/e$, presents the fluctuating electrostatic potential of the island. The CGF $S(\{\chi_i\})$ can be then represented in the form of a real-time path integral over the fields $\varphi_{1,2}(t)$ residing at two branches of the Keldysh contour

$$\exp(-S\{\chi_i\}) = \int D\varphi_{1,2}(t)\exp\Big\{\frac{i}{2}E_C^{-1}\int_{-\infty}^{+\infty}dt(\dot{\varphi}_1^2-\dot{\varphi}_2^2) - \sum_k S_{\rm con}^{[k]}(\{\hat{G},\hat{G}_k^{\chi}\}) - i\pi\delta^{-1}{\rm Tr}[(i\partial_t-\dot{\Phi})\hat{G}]\Big\}. \tag{2}$$

Here $\hat{\Phi} = {\rm diag}(\varphi_1(t), \varphi_2(t))$ is the diagonal matrix in Keldysh space; 2×2 matrix $\hat{G}(t_1,t_2)$ presents the electron Green function in the island that implicitly depends on $\varphi_{1,2}(t)$. The trace operation includes the summation over Keldysh indices and the integration in time. The contribution of each connector $S_{\rm con}^{[k]}$ has a form found in the circuit theory [16,18]

$$S_{\rm con}^{[k]} = -\frac{1}{2}\sum_n {\rm Tr}\ln\Big[1+\frac{1}{4}T_{n,0}^{[k]}(\{\hat{G},\hat{G}_k^{\chi}\}-2)\Big], \tag{3}$$

$\{\hat{G},\hat{G}_k^{\chi}\}$ denoting the anticommutator of the Green functions with respect to both Keldysh and time indices. The Green functions in the leads $\hat{G}_k(\chi)$ are obtained from the equilibrium Green functions $\hat{G}_k^{(0)}$ in the reservoir k, $\hat{G}_k^{\chi}(\epsilon) = e^{i\chi_k\bar{\tau}_3/2}\hat{G}_k^{(0)}(\epsilon)e^{-i\chi_k\bar{\tau}_3/2}$, $\bar{G}_k^{(0)} = (1-2f_k)(\tau_3 + i\tau_2) - \tau_1$, $f_k(\varepsilon)$ being the electron distribution function in the corresponding reservoir. The expression (3) is valid under assumption of instantaneous electron transfer via a connector; this corresponds to energy independent $T_{n,0}^{[k]}$.

In order to find $\hat{G}(t_1,t_2)$ at given $\varphi_{1,2}(t)$, we minimize the action (2) with respect to all $\hat{G}(t_1,t_2)$ subject to the constrain $\hat{G}\circ\hat{G} = \delta(t_1-t_2)$. The solution of the corresponding saddle point equation expresses $\hat{G}(t_1,t_2) \equiv \hat{G}(t_1,t_2;[\varphi_{1,2}(t)])$ via the reservoir Green functions $\hat{G}^{[k]}$. This procedure disregards the mesoscopic fluctuations, since those lead to corrections of the order of $\sim 1/g_0$ at all energies, whereas the interaction corrections are of the order of $\sim(1/g_0)\ln(E)$ tending to diverge at small energies. In special case $\varphi_{1,2}(t) = 0$, the saddle point equation separates in energy representation and its solution can be found in Ref. [16]. We also require $\delta \lesssim \{eV, kT\}$; i.e., our effective action (2) is applicable provided δ is the smallest energy scale in the problem.

Perturbation theory and renormalization group.—We start the analysis of the model with perturbation theory in $\varphi_{1,2}$ around the semiclassical saddle point $\hat{G}(t_1,t_2) = \hat{G}_0$, $\varphi_{1,2}(t) = 0$. The phase $\varphi(t)$ is the conjugated variable to the total charge Q on the island [17]. At large conductance, $g_0 \gg 1$, the quantum fluctuations of charge are big while $\varphi(t)$ is well defined, its fluctuations being small, $\delta\varphi^2 \sim 1/g_0$. Thus we keep only quadratic terms to the action (2). The resulting Gaussian path integral over $\varphi_{1,2}$ can be readily done. This procedure is equivalent to the summation of all one-loop diagrams of the conventional perturbation theory, i.e., to the "random-phase approximation" (RPA).

For the rest, we restrict ourselves to the most interesting low voltage-temperature limit, $\max\{eV, kT\} \ll g_0E_C$. In this limit, we evaluate the interaction correction to the CGF with the logarithmic accuracy. It reads

$$\Delta S_{\chi} = \frac{t_0}{g_0}\ln(\Lambda)\int\frac{d\varepsilon}{2\pi}\sum_{n,k}\frac{2T_{n,0}^{[k]}(1-T_{n,0}^{[k]})(\{\hat{G}_k^{\chi},\hat{G}_0\}-2)}{4+T_{n,0}^{[k]}(\{\hat{G}_k^{\chi},\hat{G}_0\}-2)}. \tag{4}$$

Here $\Lambda = g_0E_C/\max\{eV, kT, E_{\rm Th}\}$, with $E_{\rm Th} = g_0\delta$ being the Thouless energy of the island. Note that the correction (4) is contributed by only virtual inelastic processes that change probabilities of real elastic scatterings.

For simplicity, we consider the shot-noise limit $eV \gg kT$ only. Then the magnitude of the correction shall be compared with the zero-order CGF $S_{\chi}^{[0]} \sim t_0eVg_0$. This implies that the perturbative RPA result (4) is applicable only if $g_0^{-1}\ln(g_0E_C/eV) \ll 1$. At lower voltages ΔS_{χ}

logarithmically diverges. This indicates that we should proceed with a renormalization group (RG) analysis.

We perform the RG analysis of the action (2) in the one-loop approximation. This is justified by $g_0 \gg 1$. We follow the conventional procedure and decompose $\varphi(t)$ onto the fast φ_f and slow parts φ_s. On each step of RG procedure we eliminate the fast degrees of freedom in the energy range $E - \delta E < \omega < E$ to obtain a new action $S_{E-\delta E}[\varphi_s]$, E being the running ultraviolet cutoff. Our key result is that the change in the action at each step of RG procedure can be presented as a change of $T_n^{[k]}$.

Therefore, the RG equations can be written directly for $T_n^{[k]}$ and take a simple form (1). These equations are to be solved with initial conditions $T_n^{[k]}(E = g_0 E_C) = T_{n,0}^{[k]}$ at the upper cutoff energy $E = g_0 E_C$, where $T_{n,0}^{[k]}$ are transmission eigenvalues in the absence of interaction. The RG Eqs. (1) resemble those for the transmission coefficient for a scatterer in the weakly interacting one-dimensional electron gas [15] and for a single multichannel scatterer in the electromagnetic environment [13]. The effective impedance Z is just replaced by inverse conductance of the island to all reservoirs, $G(E) = G_Q \sum_{n,k} T_n^{[k]}(E)$. The important point is that this conductance is itself subject to renormalization. This difference becomes most evident in the case when all contacts are tunnel junctions, $T_n^{[k]} \ll 1$. In this case, one can sum up over k, n in Eqs. (1) to obtain the RG equation for the conductance only [19]: $dG/d\ln E = 2G_Q$. The Eqs. (1) could be also derived in the framework of functional RG approach to σ model of disordered metal [20].

The RG Eqs. (1) can be solved by introducing an auxiliary variable $y(E)$ defined by $dy(E) = dE/\sum_{n,k} T_n^{[k]}$

$$T_n^{[k]}(E) = T_{n,0}^{[k]} y/[1 - T_{n,0}^{[k]}(1 - y)],$$
$$\ln(g_0 E_C/E) = -(1/2)\sum_{n,k} \ln[1 - T_{n,0}^{[k]}(1 - y)]. \tag{5}$$

The first equation gives the renormalized transmission eigenvalues at a given value E of the upper cutoff in terms of variable $y(E)$, $0 \leq y \leq 1$. The second equation implicitly expresses $y(E)$.

We note that the energy dependence of transmission coefficients induced by interaction is very weak provided $G(E) \gg G_Q$: if energy is changed by a factor of 2, the conductance is changed by $\sim G_Q$. To use the equations for evaluation of FCS at given voltages $V^{[k]}$ of the leads, one takes $T_n^{[k]}(E)$ at upper cutoff $E = \max_k(V^{[k]})$ and further disregards their energy dependence.

Two scenarios of low-energy behavior.—The RG Eqs. (1) have a fixed point at $T_n^{[k]} = 0, y = 0$ that occur at finite energy

$$E = \tilde{E}_C = g_0 E_C \prod_{k,n} (1 - T_{n,0}^{[k]})^{1/2}. \tag{6}$$

Since all $T_n^{[k]} = 0$, this indicates the isolation of the dot from the leads, localization of charge in there, and formation of Coulomb blockade with the exponentially small gap $\tilde{E}_C$. The same energy scale was obtained from the equilibrium instanton calculation of Ref. [8]. For a field theory, one generally expects different physics and different energy scales for instantons and perturbative RG. The fact that these scales are the same shows a hidden symmetry of the model which is yet to be understood.

Alternative low-energy behavior is realized if the running cutoff reaches $E_{\rm Th} = G(E)\delta/G_Q$ (Fig. 2). The logarithmic renormalization of $T_n^{[k]}$ stops at this point and their values saturate. We thus predict a sharp crossover between two alternative scenarios that occur at value $g_0 = g_c$ corresponding to $\tilde{E}_C \simeq \delta$. This value equals $g_c = \alpha^{-1}\ln(E_C/\delta)$, with a factor α depending on transmission distribution of all connectors. If all connectors are tunnel junctions, $\alpha_T = 1/2$. For diffusive connectors $\alpha_D = \pi^2/8$ and $g_D(E) \sim g_0\sqrt{\xi}\cot\!\mathrm{an}\sqrt{\xi}$, $\xi \equiv 2g_0^{-1}\ln(g_0 E_C/E)$. As seen from Fig. 2, diffusive connectors suppress the Coulomb blockade much more efficiently than tunnel contacts of the same total conductance. For example, at $\ln(E_C/\delta) = 10$ the initial tunnel conductance $G_T^0 = 14G_Q$ [lowermost curve at Fig. 2(a)] results in Coulomb blockade at low energies while the same initial diffusive conductance just saturates to $G_D = 6$ [Fig. 2(b)].

We stress the generality of the results obtained. Since the connectors assumed to be arbitrary, eventually *any* nanostructure with conductance $G > G_Q$ subject to weak Coulomb interaction and able to store charge inside itself can be approximated by the model in use. Therefore, our results are relevant for *any* nanostructure of this kind.

For a metallic grain of size L, one estimates $E_C/\delta \sim (L/\lambda_{\rm F})^2$, $\lambda_{\rm F}$ being the Fermi wavelength. Therefore, the critical conductance $g_c \sim 2\alpha^{-1}\ln(L/\lambda_{\rm F}) \gg 1$ may be relatively large and the crossover between two scenarios can be examined experimentally. This crossover has indeed

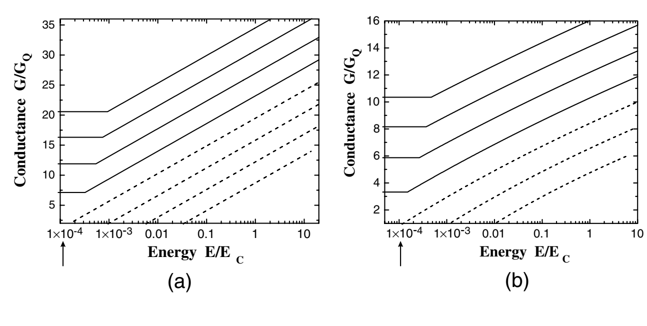

FIG. 2. The energy dependence of the total conductance either saturates (solid lines) or hits zero at Coulomb blockade gap $\tilde{E}_C$ (dashed lines). These two scenarios are presented for (a) tunnel connectors, g_0 changing from 42 to 14 with the step 4 from upper curve down; (b) diffusive connectors, g_0 changing from 18 to 6 with the step 2. Arrows indicate the crossover energy scale $\sim\delta$. We assume $\ln(E_c/\delta) = 10.0$.

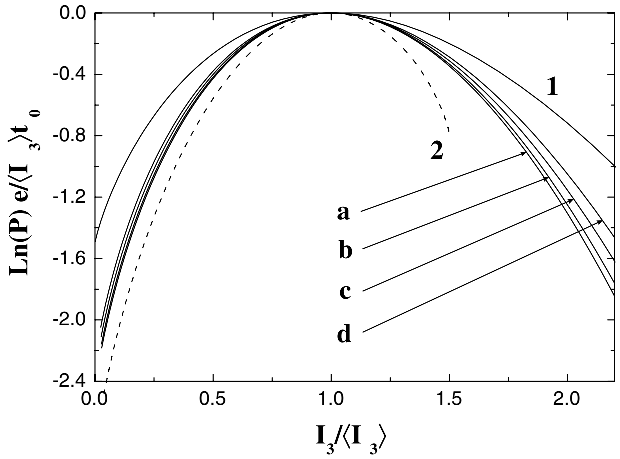

FIG. 3. The probability of big fluctuation of the current I_3 at $I_1 = I_2$ for the 3-terminal island with identical junctions (Fig. 1). Curves (a)–(d) correspond to diffusive connectors at different values of the renormalization parameter $y(eV)$ [See Eqs. (5)]: (a) 1.0, (b) 0.5, (c) 0.1, (d) 0. No interaction effect is seen for tunnel (1) and ballistic (2) connectors.

been observed in [9]. We believe that our results will facilitate further experiments in this direction.

Full counting statistics.—To give an example of FCS calculation in the regime of weak Coulomb interaction, we consider the 3-terminal Coulomb island, shown in Fig. 1, with identical tunnel ($T_n \ll 1$), ballistic ($T_n = 1$ or 0), and diffusive contacts. In diffusive contacts T_n are distributed according to the universal law $\rho_D(T) = g_D^0/2T\sqrt{1-T}$. We plot in Fig. 3 the $\log(P)$, the logarithm of the probability to measure the same currents to the terminals 1 and 2, $I_1 = I_2 = -I_3/2$, versus the current I_3 measured in the terminal 3. The voltages applied are $V_3 = V$, $V_{1,2} = 0$; $eV \gg T$. Both $\log(P)$ and I_3 are normalized by the average current $\langle I_3 \rangle$, so that in the absence of interaction the curves corresponding to different voltage are the same. We stress that the shape of these curves is determined only by the transmission distribution $\rho(T)$ in the contacts. To account for the interaction, we change $T_n^{[k]}$ according to Eqs. (5) and evaluate the probability using the method of Ref. [15].

The curves 1 (tunnel junctions) and 2 (ballistic contacts) stay the same not depending on the renormalization. Indeed, according to Eq. (1) the renormalization does not affect ballistic transmission. As to tunnel junctions, it affects only their conductances. The interaction effect is visible for diffusive junctions. The curves (a)–(d) correspond to decreasing values of $y(E = eV)$. The transmission distribution of each contact evolves from the diffusive form ($\rho_D(T) = g(y)/2T\sqrt{1-T}$) at the highest voltage ($y \approx 1$) to the double junction form ($\rho_{DJ}(T) = g(y)/\pi T^{3/2}\sqrt{1-T}$) at the lowest voltage ($y \approx 0$) [13]. Since the normalized probability distribution reflects $\rho(T)$, this visibly changes its current dependence.

To conclude, we have analyzed the effect of weak Coulomb interaction ($G \gg G_Q$) on FCS in Coulomb island. The interaction effect can be incorporated into an energy-dependent renormalization of transmission eigenvalues; this enables easy evaluation of all transport properties. The Coulomb blockade develops only if the "high-voltage" total conductance of the island is below a critical value $\sim G_Q \log(E_C/\delta)$, otherwise the interaction correction to transport saturates at low energies.

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