Structure-Based Interpretation of the Strouhal-Reynolds Number Relationship

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The wake in the flow past a circular cylinder has posed a long-standing challenge to scientists since the late 19th century. Many aspects of this seemingly simple phenomenon remain unexplained. Of particular interest is the relationship between the dimensionless vortex shedding frequency (the Strouhal number St) and the ratio of inertial to viscous forces in the fluid (the Reynolds number Re). We propose a new St-Re relation based on the observations of the structure of a vortex street in flowing soap films. The measurements suggest a simple two-parameter form $\mathrm{St} = 1/(A + B/\mathrm{Re})$ that describes laminar vortex shedding remarkably well for bulk fluids as well as for two-dimensional flowing soap films.

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Vortex shedding from a bluff body is a common phenomenon and has been studied since Strouhal's pioneering work [1]. However, not all aspects of vortex shedding are well understood. One of the long-standing problems is the precise relationship between the shedding frequency f and a linear dimension of the bluff body, say, the diameter D of a circular cylinder, and the mean velocity U [2]. Motivated by the idea of hydrodynamic similarity, Rayleigh proposed that if one defines a dimensionless frequency $\mathrm{St}(\equiv fD/U)$, termed the Strouhal number, St is related to the Reynolds number $\mathrm{Re}(\equiv UD/\nu)$ as

$$\mathrm{St} = a\left(1 - \frac{b}{\mathrm{Re}}\right), \tag{1}$$

where ν is the kinematic viscosity of the fluid and a and b are constants [3]. Although measurements have largely supported Rayleigh's proposition with small modifications to the coefficients a and b [4], systematic deviations to Eq. (1) were noticeable near the onset of vortex shedding [5]. These deviations have motivated current research with the aim of finding a better description of the St-Re relation [6,7]. Various new scaling relations were proposed, some adding correction terms to Eq. (1) and others conjecturing entirely different forms [5]. Among various proposals, the most prominent one is the relation given by Fey *et al.* [6] and Williamson and Brown [7]:

$$\mathrm{St} = a' - \frac{b'}{\mathrm{Re}^{1/2}}. \tag{2}$$

The primary justifications for Eq. (2) are that it fits experimental data well and it may be derivable from a boundary layer theory [7].

Herein we report new investigations of the St-Re relation in the flowing soap film channel with the focus on the structure of the vortex street. An important feature of flowing soap film is its small thickness, ensuring that the velocity is confined to the film plane and thus two dimensional (2D). This property eliminates a variety of instabilities that occur in bulk fluids, complicating the interpretation of data. We note that in three-dimensional (3D) fluids, Eq. (1) is only piecewise applicable in different Re regimes and works the best in the laminar shedding regime ($50 < \mathrm{Re} < 180$) [5]. In contrast, vortex streets in the soap film are stable for Re up to several thousand, allowing St-Re relation to be tested over a much broader range of Re. By definition, the shedding frequency downstream is given by $f = v_{\mathrm{st}}/\lambda$, where v_{st} and λ are the street velocity and the spatial periodicity in the laboratory frame, and both are functions of U and D. In this experiment, v_{st} and λ were measured as a function of D while keeping U fixed. We found that for a wide range of D (and consequently Re) the street velocity v_{st} normalized by U is constant $c(= v_{\mathrm{st}}/U) \sim 0.75$ for $70 < \mathrm{Re} < 3000$. We also found that λ depends linearly on D (or Re), $\lambda = \lambda_0 + \alpha D$. These relations are surprisingly simple considering the complexity of the phenomenon. Imposing the hydrodynamic similarity hypothesis, it follows that

$$\mathrm{St} = \frac{1}{A + B/\mathrm{Re}}, \tag{3}$$

where $A = \alpha/c$ and $B = \lambda_0 U/(c\nu)$. This equation not only describes our own data very well for more than two decades in Re but also fits well-known modern 3D measurements published in literature [5,8]. A small correction arises near the onset of vortex shedding, but it is hardly detectable unless the data are of exceptional quality.

The measurements were carried out in a flowing soap film channel 10 cm wide and 3 m long. The flow in the film is laminar and can last indefinitely because of a built-in recirculation system (see Ref. [9] for details). To generate vortex streets, tapered glass rods were used, which allowed Re to be varied continuously without changing U. By fixing $U(= 120\ \mathrm{cm/s})$, both the film thickness h and its viscosity ν were held constant during measurements. Based on a previous experiment of comparable conditions, it is determined that $h \approx 3\ \mu\mathrm{m}$ and $\nu \approx 2 \times 10^{-2}\ \mathrm{cm^2/s}$ [9]. By pulling the rod slowly out of the film, D continuously decreased, permitting the determination of the onset of vortex shedding. We found that the onset of the vortex shedding $\mathrm{Re}_C = 11 \pm 2$ was consistently lower than the 3D value of $\mathrm{Re}_C = 47$ with a negligible hysteresis [10].

Using a fast video camera, vortex shedding frequency f was measured directly by counting the number of vortices peeled off from the rod. The measured St versus Re is plotted in Fig. 1. We note that the asymptotic Strouhal number ($\mathrm{Re} \to \infty$) $\mathrm{St}_\infty = 0.195$ is $\sim$10% lower than the typical 3D measurements [5] or 2D measurements in films [11]. Nevertheless, this difference does not alter our conclusion that Eq. (3) is an accurate representation of vortex shedding from circular cylinders. Since St is also given by $(v_{\mathrm{st}}/U)(D/\lambda)$, a careful study of the dependence of v_{st} and λ on Re can yield useful information about the St-Re relation. Our imaging technique allows us to locate the centers of the vortices and to measure the distances between pairs of vortices on each side of the rod, which we called $\lambda(x)$. The street velocity $v(x)$ was also measured at different downstream locations x. As shown in Fig. 2(a), both $\lambda(x)$ and $v(x)$ were found to increase with x and eventually reach the steady-state values λ_∞ and v_∞. These measurements showed that the spatial dependence of $\lambda(x)$ and $v(x)$ is nearly identical, suggesting that vortex shedding frequency $f = v(x)/\lambda(x) = v_\infty/\lambda_\infty$ is independent of x. This property supports the view that the Karman vortex street is a single global mode characterized by the frequency f. Since we are only interested in the steady state, in what follows λ_∞ and v_∞ will be replaced by λ and v_{st} for simplicity of notation.

In Fig. 2(b) the steady-state values of λ and v_{st} are plotted against Re. We found that to a good approximation λ is a linear function of D (or Re) and can be parametrized as $\lambda = \lambda_0 + \alpha D$, where $\lambda_0 = 0.35 \pm 0.08$ mm and $\alpha = 4.1 \pm 0.3$. Since $\lambda_0 > 0$, the wake periodicity does not vanish in the limit $\mathrm{Re} \to \mathrm{Re}_C$ and λ_0 represents the smallest unstable wavelength in the system. By introducing the dimensionless wavelength $\lambda_D \equiv \lambda U/\nu$, this relation can be rewritten as $\lambda_D = 210 + 4.1\mathrm{Re}$, which is presented in the inset of the figure. The normalized street velocity v_{st}/U exhibits a more complex Re dependence; it is unity for $\mathrm{Re} \to \mathrm{Re}_C$, indicating that near Re_C the wake travels at the same velocity as the mean velocity U. As Re increases, v_{st}/U decreases rapidly and saturates at $c \approx 0.75$. This value is about 13% lower than c observed in 3D flows [12], indicating that more vorticity is encapsulated into the vortex in flowing soap films. The nontrivial Re dependence

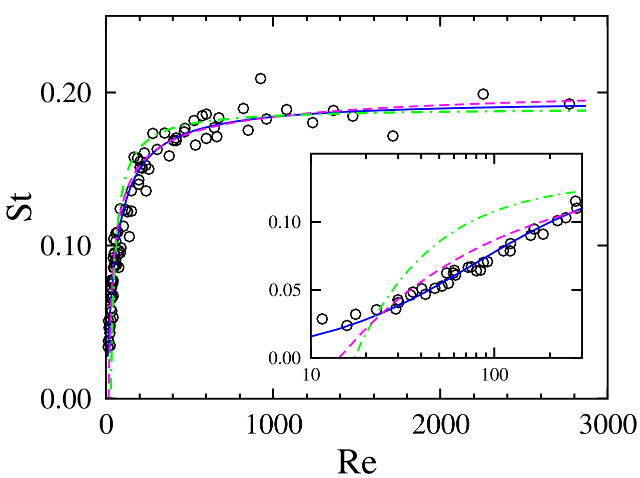

FIG. 1 (color online). The measured St versus Re relation. The main curve represents measurements carried out in the vertical film, while the data in the inset are for an inclined film. The solid lines are fits by using $\mathrm{St} = 1/(A + B/\mathrm{Re})$. This new relation out performs Eq. (1) $\mathrm{St} = 0.18(1 - 28/\mathrm{Re})$ (dash-dotted line) and Eq. (2) $\mathrm{St} = 0.21 - 0.825/\mathrm{Re}^{1/2}$ (dashed line). The difference becomes more significant near Re_C as delineated in the inset.

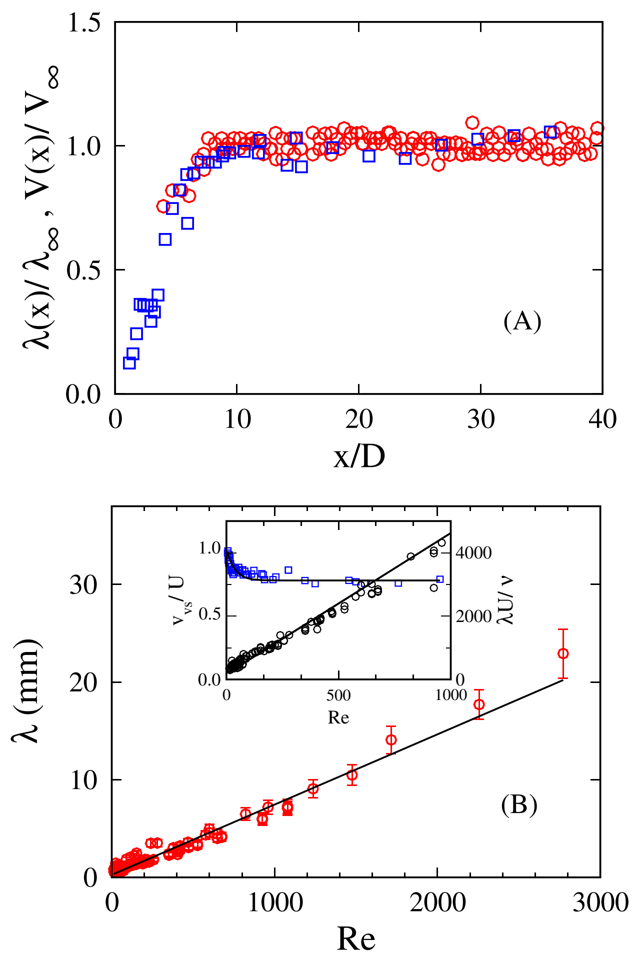

FIG. 2 (color online). (a) The wavelength $\lambda(x)$ and the wake velocity $v(x)$ of a Karman vortex street versus downstream distance x. The measurements were carried out using a rod of diameter $D = 0.056$ cm and $U = 70$ cm/s. Both the wake velocity $v(x)$ (squares) and the wavelength $\lambda(x)$ (circles) reach their saturation values $\lambda_\infty = 0.225$ cm and $v_\infty = 53$ cm/s rapidly, within $\sim 7D$ downstream. (b) The steady-state wavelength λ and street velocity v_{st} versus Re. Using different D's, dependence of $\lambda = \lambda_\infty$ and $v_{\mathrm{st}} = v_\infty$ on Re was measured. It can be seen, while λ is approximately linear in Re, the street velocity (squares) is strongly nonlinear near Re_C. This cusp can be modeled approximately by a function that has a $\mathrm{Re}^{1/2}$ dependence, as delineated by the solid line. The dimensionless wavelength $\lambda U/\nu$ (circles) is also shown in the inset.

near Re_C is expected if the onset is a Hopf bifurcation, since in this case the transverse component of the velocity vanishes continuously as $(\mathrm{Re} - \mathrm{Re}_C)^{1/2}$ [13]. In our experiment, the behavior of v_{st} near Re_C can be adequately described as $v_{\mathrm{st}}/U = c + (1 - c)f(\mathrm{Re})$, where $f(\mathrm{Re})$ has the property that it is unity when $\mathrm{Re} = \mathrm{Re}_C$ and it vanishes for $\mathrm{Re} \gg \mathrm{Re}_C$. Our measurements support a power-law dependence $f(\mathrm{Re}) = (\mathrm{Re}_C/\mathrm{Re})^{0.5\pm0.3}$ as delineated by the solid line in the inset of Fig. 2(b). This power-law term improves the quality of the fit but not to the extent that it could not be neglected.

The measured Re dependences for v_{st} and λ provide an alternative means for obtaining the St-Re relation:

$$\mathrm{St} = \left(\frac{v_{\mathrm{st}}}{U}\right)\left(\frac{D}{\lambda}\right) = [c + (1 - c)f(\mathrm{Re})]\left(\frac{D}{\lambda_0 + \alpha D}\right), \quad (4)$$

and its agreement with the direct frequency measurement is reassuring. What is surprising, however, is that even if the "critical" anomaly $f(\mathrm{Re})$ is neglected, the simplified relation $\mathrm{St} = 1/(A + B/\mathrm{Re})$ still works remarkably well for our system. Such a good agreement arises because the critical regime is narrow and the data are not noise free, allowing the two-parameter equation to be extrapolated into the critical regime. The solid lines in Fig. 1 illustrate the quality of fit over two decades in Re using Eq. (3). As can be seen, it describes the data much better than the classical relation Eq. (1) and is considerably better than Eq. (2). The difference is striking for the low-Re regime as presented in the inset. Further comparison is presented in Table I.

We next turned our attention to the implication of our new St-Re relation on previous measurements, particularly those in 3D fluids. Modern measurements in the laminar shedding regime ($47 < \mathrm{Re} < 180$) typically have errors of a few percent and the discrepancy between different measurements is also of the same magnitude [5]. The only complication is that in 3D fluids the measured St-Re relation is piecewise continuous; vortex shedding is laminar only for the above indicated range of Re. Our discussion will be focused in this regime. We note that for this narrow range of Re the fits using our scheme and that of Eq. (2) are not significantly different. The quality of the fits can be best demonstrated by plotting the residuals between the data and the fit, which are delineated by differently colored lines as shown in Fig. 3. The quality of the fit can also be judged by the χ^2 (see Table I). We note that in all cases Eq. (1) is the least accurate presentation of the measurements. On the other hand, both Eqs. (2) and (3) yield comparable fits with the difference $\chi^2 \sim 10^{-6}$ that may not be significant. Recall that Eq. (3), containing two parameters, is only an approximation with the critical anomaly being ignored. Including such an anomaly could improve the quality of fit near Re_C, but such an inclusion may not be meaningful considering noises in the data.

As most hydrodynamic problems, it is of interest to explore the asymptotic behavior of the system. We note that for large Re, the denominator in St can be expanded in terms of $1/\mathrm{Re}$ as Rayleigh suggested [3]. In increasing orders of $1/\mathrm{Re}$, one finds from Eq. (4)

$$\mathrm{St} = \frac{1}{\alpha}\left[c + (1 - c)\left(\frac{\mathrm{Re}_C}{\mathrm{Re}}\right)^{1/2} - \left(\frac{c}{A}\right)\left(\frac{B}{\mathrm{Re}}\right) - \left(\frac{1 - c}{A}\right)\left(\frac{B}{\mathrm{Re}}\right)\left(\frac{\mathrm{Re}_C}{\mathrm{Re}}\right)^{1/2} + \cdots\right]. \quad (5)$$

This equation also contains the $1/\mathrm{Re}^{1/2}$ term proposed by Fey *et al.* [6] and Williamson and Brown [7]. However, for practical considerations, it can be shown that the leading term is $1/\mathrm{Re}$ instead of $1/\mathrm{Re}^{1/2}$. Comparing the second and third terms in Eq. (5), it is evident that for the $1/\mathrm{Re}^{1/2}$ term to be dominant $\mathrm{Re} > [c/(1 - c)]^2(B/\mathrm{A})^2/\mathrm{Re}_C$ or $\mathrm{Re} > 3000$ for 2D films. Thus, for our experiment the contribution of the second term is negligible, validating our early observation that the critical anomaly $f(\mathrm{Re})$ can be neglected. One can draw the same conclusion for laminar shedding in bulk fluids.

To summarize, we have found a new St-Re relation, which is firmly based on the observations of the structure of a vortex street and its motion over a broad range of Re. Although the behavior of vortex streets in the soap films and those in incompressible 3D fluids are somewhat different, possibly due to the film being slightly compressible, our measurements suggest a simple two-parameter form $\mathrm{St} = 1/(A + B/\mathrm{Re})$ that describes remarkably well our data as well as the 3D data for circular cylinders. Since this relation is based on simple observations that are expected to be general, we postulate that Eq. (3) may be applicable to other slender bodies. Finally, we point out that the new relationship $\mathrm{St} = 1/(A + B/\mathrm{Re})$ appears similar to the classical relation $\mathrm{St} = a(1 - b/\mathrm{Re})$ proposed by Rayleigh. In his original proposition, the $1/\mathrm{Re}$ dependence is simply the leading term of an infinite expansion in terms of the control parameter Re. Indeed, for large Re the two relations are identical with the result $a = 1/A$ and

TABLE I. Fitting results.

	Experiments	A	B	χ^2 of Eq. (1)	χ^2 of Eq. (2)	χ^2 of Eq. (3)
2D	Flowing films (this work)	5.12	313	1.5×10^{-3}	2.4×10^{-4}	1.1×10^{-4}
2D	Henderson [14]	4.03	202	5.0×10^{-6}	1.4×10^{-6}	1.6×10^{-6}
3D	Williamson [5]	4.18	193	7.3×10^{-6}	1.3×10^{-6}	5.0×10^{-7}
3D	Norberg [8]	4.15	197	4.1×10^{-6}	3.9×10^{-7}	9.0×10^{-7}

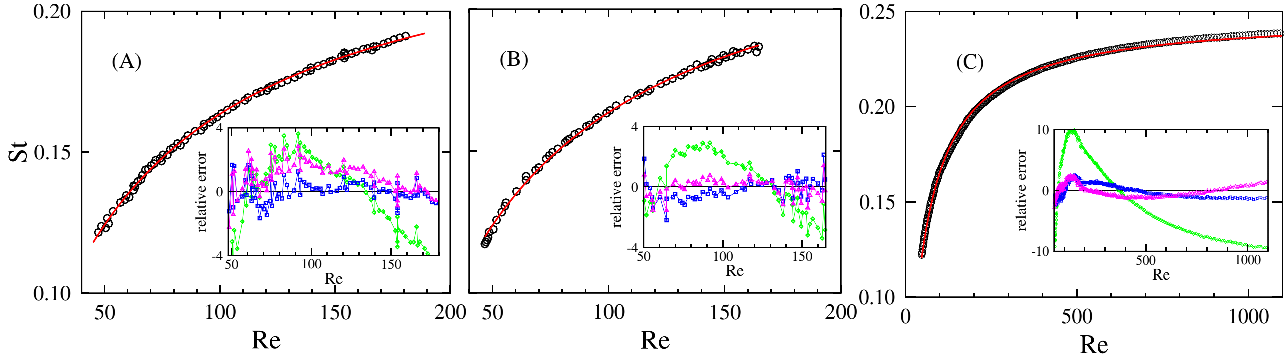

FIG. 3 (color online). The fits to 3D experimental data and 2D numerical simulations. (a),(b) 3D measurements were carried out by Williamson [5] and Norberg [8], respectively. (c) In Henderson's numerical simulations, since secondary 3D instabilities are absent, data are continuous up to $\mathrm{Re} \sim 1000$ [14]. The solid lines in all three plots present fittings by using Eq. (3). Insets are plots of residuals between the fit and the data, using Eq. (1) (circles), Eq. (2) (triangles), and Eq. (3) (squares).

$b = B/A$. However, the expansion is not valid near the onset of shedding. It did not escape our attention that if we rewrite our relation in the form $\mathrm{St} = \mathrm{St}_\infty/(1 + \mathrm{Re}_0/\mathrm{Re})$, $\mathrm{Re}_0 = B/A = 47.2 \pm 0.8$ turns out to be surprisingly close to the critical Reynolds number $\mathrm{Re}_C = 47 \pm 3$ measured in 3D fluids. Here A and B are taken from the fits to the data of Williamson and Norberg. Thus, the new St-Re relation implies the existence of a special Reynolds number Re_0, which in 3D fluids can be identified as Re_C. It comes as no surprise therefore that for 3D fluids, the $1/\mathrm{Re}$ expansion of Eq. (3) converges poorly near Re_C, explaining persistent deviations from Eq. (1) when compared with modern measurements and numerical simulations (see Table I). For 2D flows in soap films, such an expansion is not possible for $\mathrm{Re} < \mathrm{Re}_0$, since $\mathrm{Re}_0 > \mathrm{Re}_C$.

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