# Detecting Vacuum Entanglement in a Linear Ion Trap

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We propose and study a method for detecting ground-state entanglement in a chain of trapped ions. We show that the entanglement between single ions or groups of ions can be swapped to the internal levels of two ions by sending laser pulses that couple the internal and motional degrees of freedom. This allows us to entangle two ions without actually performing gate operations. A proof of principle of the effect can be realized with two trapped ions and is feasible with current technology.

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A remarkable phenomenon that appears naturally in quantum field theory is that the ground state (vacuum) is entangled and that observables in two separated regions can be entangled. Recent studies in quantum information theory have taught us that entanglement is a physical property which can be exchanged between systems or used in quantum processes such as quantum communication, teleportation, and quantum cryptography [1]. This suggests that vacuum entanglement as well could be detected and used in quantum processes. There have been several studies of vacuum entanglement in field theory [2,3], as well as in other systems [4–6], but none have proposed a way to observe vacuum entanglement in a realistic experiment. The main purpose of this Letter is to suggest a realistic physical implementation to observe this phenomenon.

A gedanken experiment that allows the observation of vacuum entanglement in field theory has been suggested [3], which utilizes two basic ingredients of relativistic field theory and quantum information: the presence of a *causal structure,* and the nonincrease of entanglement under local operations; operations performed at two causally disconnected regions do not increase the entanglement between these regions. Consider two atoms, $A$ and $B$, which locally interact with the field and with one another through the long range field interaction. The interaction with the field can entangle $A$ and $B$ either via the exchange of propagating quanta or by transporting vacuum entanglement into the atoms. Using the fields' built-in causal structure, one can eliminate the former unwanted process, by demanding that $cT < L$, where $T$ is the interaction time and $L$ is the separation between the atoms. Vacuum entanglement can then be ''swapped'' to the atoms' internal levels, which can then be used for detecting vacuum entanglement. However, this method requires precise control of the atom-field interaction, which, in the case of an electromagnetic field, renders the experiment highly unrealistic. Nevertheless, experimentalists in atomic physics are taming their systems at the quantum level and can test quantum effects with the required precision and control for this type of experiment.

In the following, we propose and analyze the possibility of observing vacuum entanglement with trapped ions (Fig. 1). We consider a system of trapped ions that are brought to equilibrium in a linear chain configuration. The ground state (vacuum) of the system is an entangled state of the different motional modes of the chain and manifests entanglement between single ions or distant groups of ions. In order to detect vacuum entanglement, we consider processes wherein the external motional degrees of freedom are mapped to the internal ions states, which are then used for entanglement detection. The internal levels are well isolated, they can be temporarily coupled ''on demand'' to the phonon modes by sending finite duration laser pulses, and finally can be measured with nearly perfect precision. In analogy with the field-theoretical case, the interaction must be limited to a duration shorter than the time it takes for perturbations to propagate between the two (probe) ions along the chain. We comment that in the case of ion chains, this process is interesting on its own, because one can entangle the internal levels of two ions without actually

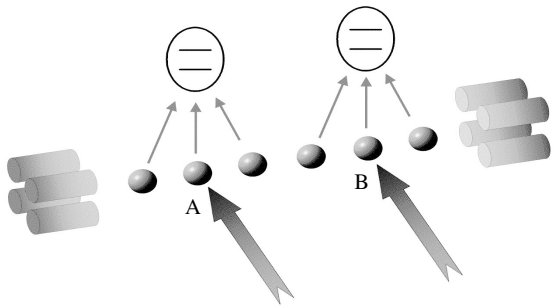


FIG. 1. Scheme for detecting ground-state entanglement: the entanglement between two groups of ions is swapped to the internal level of ions $A$ and $B$ by sending separate laser pulses that induce an interaction between the internal levels and the position of each ion.

 

doing gates [7]. The most spectacular manifestation of the idea would involve a chain with many ions. However, a proof of principle can be attained with just two trapped ions and is feasible using current technology. We study both cases and analyze the latter in detail.

We now consider a system of $N$ ions trapped in a linear Paul trap at very low temperature [8]. The Hamiltonian describing the ions' motion relative to their equilibrium positions and internal levels is $H_0 = \frac{1}{2}\omega_z(\sigma_z^{(A)} + \sigma_z^{(B)}) + \sum \nu_n a_n^\dagger a_n$. Here $\omega_z$ is the internal level energy gap of the two relevant (probe) ions $A$ and $B$, and $\nu_n$ are the phonon normal-mode frequencies, with corresponding creation (annihilation) operators $a_n^\dagger$ ($a_n$). Typically, $\omega_z$ is in the optical region and $\nu_n \sim$ MHz.

We begin by analyzing the simplest case with just two trapped ions. The vacuum state is then defined as the ground state of the normal modes of the system, i.e., a product state of the collective and breading modes $|0_c\rangle$ and $|0_b\rangle$. In terms of the *local* single oscillator states $|n\rangle_{A,B}$, the vacuum is an entangled two mode squeezed state [9]

$$|\mathrm{vac}\rangle = |0_c\rangle|0_b\rangle = \sqrt{1 - e^{-2\beta}}\sum_n e^{-\beta n}|n\rangle_A|n\rangle_B. \quad (1)$$

The local number states are the single ion energy eigenstates obtained when the displacement of the other ion is set to zero. We get $e^{-\beta} = \sqrt{(\lambda - 1/2)/(\lambda + 1/2)}$, where $\lambda = (1/4)[\sqrt{\nu_0/\nu_1} + \sqrt{\nu_1/\nu_0}]$, and $\nu_0$ and $\nu_1$ are the frequencies of the collective and breathing modes. Since $\nu_0/\nu_1 = \sqrt{3}$, we get $\lambda = 0.5189$, and the von Neumann entanglement [10] of the squeezed state is $E = (\lambda + 1/2)\log_2(\lambda + 1/2) - (\lambda - 1/2)\log_2(\lambda - 1/2) = 0.136\, e$-bit.

In order to transfer vacuum entanglement into the ion internal states, we use laser pulses to couple the internal and motional states of atoms $A$ and $B$. Close to resonance, with $\omega_{\mathrm{laser}} \approx \omega_z$, the interaction terms for the $k$th ion (in the Lamb-Dicke limit) is given by [11]

$$H_{\mathrm{int}}^{(k)} = \Omega(t)(e^{-i\phi}\sigma_+^{(k)} + e^{i\phi}\sigma_-^{(k)})x_k, \quad (2)$$

where $\sigma_\pm$ are the raising and lowering operators, $\phi$ is the laser phase, and $x_k$ is the displacement of the $k$th ion. Above we have applied the rotating wave approximation with respect to the internal levels but not (as is usually done) to position operators. That is because the duration $T$ of the laser pulses satisfies $1/\omega_z \ll T \leq 1/\nu_0$. The upper bound on $T$ follows from the requirement that perturbations do not propagate between the ions during the interaction.

Using the available interaction (2), we would like to swap the motional entanglement into the internal ion levels which are initially prepared in a nonentangled state $|\downarrow\rangle_A|\downarrow\rangle_B$. The smallness of $e^{-\beta}$ implies that the entanglement arises mostly from the first two terms of (1). We therefore seek a procedure, of typical duration $T \ll 1/\nu_0$, that maps

$$|\mathrm{vac}\rangle|\downarrow\rangle|\downarrow\rangle \rightarrow |\chi\rangle[|\downarrow\rangle|\downarrow\rangle + e^{-\beta}|\uparrow\rangle|\uparrow\rangle], \quad (3)$$

where $\chi$ is the final state of the ions. The interaction then acts separately on each ion and swaps the lowest two motional states $|0\rangle$ and $|1\rangle$ to the two ion internal states. In this $4 \times 4$ subspace the map is expected to approximate the unitary swap transformation $e^{i\pi/4(\tilde{\sigma}_x\sigma_x + \tilde{\sigma}_y\sigma_y)}$, where $\tilde{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\tilde{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$ act on the number states. (We have ignored a trivial $\tilde{\sigma}_z\sigma_z$ term.) We note that $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ can be approximated by $x$ and $p$, respectively. Based on this intuition we proceed with the following construction. We consider the following sequence of unitary operations, $U_s = V(\alpha_1)W(\beta_1)V(\alpha_2)W(\beta_2)\cdots V(\alpha_n)W(\beta_n)$, where

$$V(\alpha) = e^{i\alpha\sigma_x x}, \qquad W(\beta) = e^{i\beta\sigma_y p}, \quad (4)$$

to be performed on each ion separately by sending a sequence of laser pulses. The $V(\alpha)$ evolution can be obtained by sending a laser pulse of duration $T$ and phase $\phi = 0$, such that $T \ll 1/\nu_0$ and $\int \Omega(t)dt = \alpha$. In order to generate a $W(\beta)$ evolution, we set the laser phase to $\phi = \pi/2$ and allow the system to evolve freely for a short time interval $dt = \tau$ in between a pair of pulses. Denoting $V'(\beta) = \exp(i\beta\sigma_y x)$, we obtain

$$\begin{aligned} V'_{t=\tau}(-\beta')V'_{t=0}(\beta') &= e^{-i\{\beta'\sigma_x[x + (p\tau/m)] + O(\nu^2\tau^2)\}}e^{i\beta'\sigma_x x} \\ &= e^{-i(\beta'/m)[\sigma_x p\tau + (1/2)\tau\beta']} + O(\nu^2\tau^2), \end{aligned} \quad (5)$$

where we have used the approximation $x(\tau) = x(0) + p(0)\tau/m + O(\nu^2\tau^2)$. (Alternatively, in the Schrödinger picture we notice that $V^\dagger e^{-iH_{\mathrm{phonon}}t}V$ shifts the kinetic term as $p^2 \rightarrow (p + \beta'\sigma_y)^2 = p^2 + 2\beta'\sigma_y p + \beta'^2$.) Taking the limit $\nu^2\tau^2 \ll 1$, and maintaining $\beta \equiv \beta'\tau/m = O(1)$, we obtain the required effective coupling to $p$. Therefore, the sequence of $n$ pairs of $V(\alpha)W(\beta)$ pulses can be generated by $3n$ ordinary pulses with $n$ free evolution intermediate intervals of total duration $dt = n\tau$. By optimizing the entanglement of formation [12], $E_F(\alpha_i, \beta_i)$, over the free parameters $\alpha_i$ and $\beta_i$ the transformation (3), where $\chi$ is the final motional state of the ions, can be generated with high efficiency. This transformation swaps the first two terms in Eq. (1) to the ion's internal level states. After a sequence of three $VW$ pulses, the entanglement of formation of the final internal level state contains 97% of the computed ground-state entanglement. The optimal sequence is in this case $V(0.31)W(0.38)V(0.50)W(0.39)V(0.53)W(0.16)$. (With two pulses we get at most 93%.) Expressed in the relevant $4 \times 4$ subspace, this unitary operation has indeed a a structure which closely resembles a swap. Testing the purity of the final state $\rho_{AB}$, we find $\mathrm{tr}\,\rho_{AB}^2 = 0.997$. The final density matrix of the internal levels is depicted in Fig. 2. The measurement precision for the density matrix in recent experiments is about 1% [13] and hence sufficient for observing the entanglement of the final state.

We comment that an alternative to the above approach could be to separate the two ions by moving them apart.

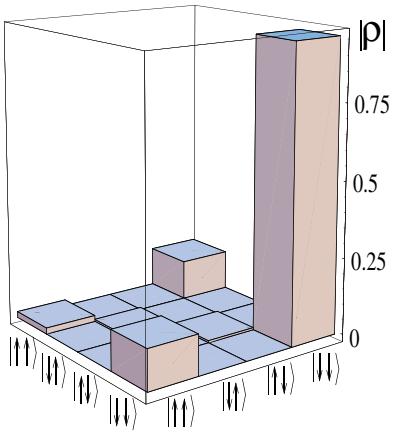


FIG. 2 (color online). Histogram of the final density matrix of the internal levels of ions $A$ and $B$. The entanglement of formation of this state accounts for 97% of the computed initial ground-state motional entanglement.

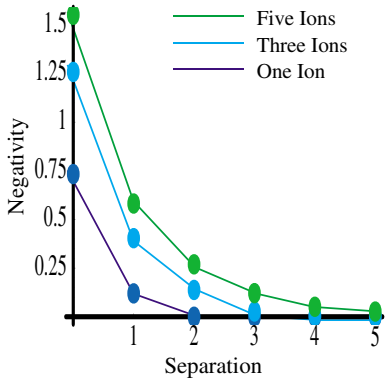


FIG. 3 (color online). Logarithmic negativity between two groups consisting of 1, 3, and 5 ions, as a function of their separation in a chain of 20 ions.

This effectively "turns off" the interaction between them, allowing for a longer duration of the detection process. It is then easier to generate the desired swap using continuous on-resonance laser pulses [14]. In order not to affect the entanglement between the ions, the separation has to be fast compared with the propagation time scale $1/\nu_0$. This can be done by increasing the potential between the ions. The possibility of changing the local potential and moving ions, without effecting the internal ion states, has been recently demonstrated experimentally [15,16].

We next turn to the more general case of entanglement in a chain with $N$ ions. The state of two *subgroups*, $\tilde{A}$ and $\tilde{B}$, each consisting of $n_A$ and $n_B$ ions, separated by $l_s$ ions, is described by a reduced Gaussian density matrix $\rho_{\tilde{A}\tilde{B}} = \mathrm{tr}_{l\tilde{A},\tilde{B}}(|\mathrm{vac}\rangle\langle\mathrm{vac}|)$. The entanglement between the groups (Fig. 3) can be characterized by the negativity [17,18]. It vanishes for separation larger than one. However, as the group size increases, it persists for larger separations.

We examine the possibility of detecting the entanglement by coupling to the internal levels of two ions, $A$ and $B$, for a time duration $T$. We can check that as long as $T < 1/\nu_0$, the interaction with respect to two complementary parts of the chain can be regarded as local. This is seen by noticing that the evolution operator, $U(T)$, can be factorized in the interaction picture as

$$U = U_A \otimes U_B \otimes e^{-i/2 \int dt dt' f(t,t') \sigma_{AB}}, \tag{6}$$

where $U_k$ act on $A$ or $B$, $\sigma_{AB} \equiv \Pi_k (e^{-i\phi}\sigma_+^{(k)} + e^{i\phi}\sigma_-^{(k)})$, and $f(t-t') = [x_A(t), x_B(t')]$. The last term above, involving $\sigma_{AB}$, is a unitary operator that can increase entanglement "nonlocally." However, as can be seen in Fig. 4, the noncommutativity described by $f(t-t')$ vanishes rapidly, and for sufficiently short interaction time, or large enough spatial separation, this noncausal effect is suppressed.

We begin with the initial ground state $|\mathrm{vac}\rangle|\downarrow\rangle|\downarrow\rangle$ and proceed to evaluate the reduced state $\rho_{AB}(T) = \mathrm{tr}[U(T)|I\rangle\langle I|U^\dagger(T)]$ perturbatively. Assuming that the intensity of the laser pulses is sufficiently weak, we expand $U(T)$ in a power series, and to lowest order in $\Omega$ we obtain

$$\rho_{AB} = \begin{pmatrix} \|X_{AB}\|^2 & 0 & 0 & -\langle 0|X_{AB}\rangle \\ 0 & \|E_A\|^2 & \langle E_B|E_A\rangle & 0 \\ 0 & \langle E_A|E_B\rangle & \|E_B\|^2 & 0 \\ -\langle X_{AB}|0\rangle & 0 & 0 & 1 - \|E_A\|^2 - \|E_B\|^2 \end{pmatrix}. \tag{7}$$

Here $|E_A\rangle = X_A|\mathrm{vac}\rangle$, $|X_{AB}\rangle = X_A X_B|\mathrm{vac}\rangle$, $X_k = \int dt \Omega(t) e^{i\delta t} x_k(t)$ $(k = A, B)$, and $\delta$ is the detuning. Using the Peres-Horodecki separability criterion, it then follows that $\rho_{AB}(T)$ is entangled if and only if $\mathcal{N}(\rho_{AB}) \approx |\langle 0|X_{AB}\rangle| - \|E_A\|\|E_B\| > 0$, where $\mathcal{N}(\rho_{AB})$ is the negativity. This condition can be understood physically as the requirement that the virtual off-shell single phonon exchange process (described by $X_{AB}$) is sufficiently large to overcome the decoherence effects due to local phonon emission (described by the $E_{A,B}$ terms).

To verify that the above condition amounts to a detection of vacuum entanglement, rather than a direct interaction due to the nonlocal correction in Eq. (3), we have repeated the computation, using the same initial state $|I\rangle$, but with a modified truncated evolution. In the latter truncated case, we "disconnected" the chain by eliminating the interaction between ions at the different halves, $n > N/2$ and $n < N/2$, of the chain. This can be easily achieved by replacing the potential term in the free phonon Hamiltonian by $\sum x_i G_{ij} x_k \to \sum x_i G^T_{ij} x_k$, where $G^T = G_A \oplus G_B$ is block diagonal. This truncated evolution does not change the entanglement between the two halves of the chain since an exact separability holds in Eq. (3) (i.e., $f(t-t') = 0$). The ratio $\eta = |\langle 0|X_{AB}\rangle|/\|E_A\|\|E_B\|$ is plotted in Fig. 5 as a

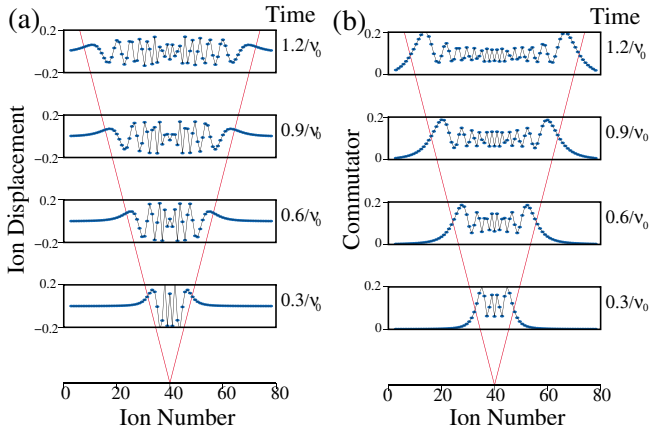


FIG. 4 (color online). (a) Classical propagation of a perturbation originating at the center of the chain. (b) The commutation relation between the displacement operators of the $n$th ion and the central ion in a chain of 80 trapped ions, at different time slices.

function of the detuning $\delta$, for $N = 20$ ions. In Fig. 5(a) the probe ions are situated at sites $l_A = 6$ and $l_B = 15$, and we can see that $A$ and $B$ become entangled ($\eta > 1$) in a range of frequencies. Since small violations of causality are expected in the nontruncated model, one could have anticipated that the nontruncated case should give rise to more entanglement. On the contrary, we see that it is the truncated case which gives rise to more entanglement. To understand this consider first Fig. 5(b), in which nearest neighbor ions $l_{A,B} = 10, 11$ have been used as probes. We find that the truncated and nontruncated models precisely agree for sufficiently small $T \ll 1/\nu_0$. In this case, since there is preexisting local entanglement between the close ions, the probe can detect entanglement in an arbitrarily short time, and the truncated interaction has here no effect because evolution is not required. On the other hand, in the

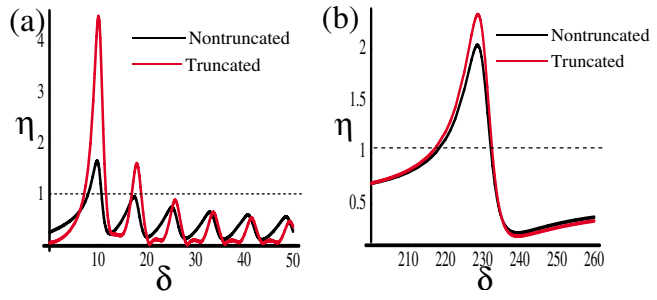


FIG. 5 (color online). The ratio $\eta$ for ions in a chain of $N = 20$ ions as a function of the detuning $\delta$. (a) The ion probes are located at $l = 6, 15$ and $T = 0.8$ (in units wherein $\nu_0 = 1$). (b) $l = 10, 11$ and $T = 0.05$. The range $\eta > 1$ signifies entanglement.

case of separated ions, propagation effects "communicate" between the probes and ions closer to the center of the trap which carry the most entanglement. This suggests that the larger entanglement in the truncated model is due to perfect wave reflection at the boundary between the regions.

In conclusion, we have proposed an efficient method for detecting vacuum entanglement by mapping motional states of trapped ions or groups of ions to the ions' internal levels. It is remarkable that this phenomenon, which can be considered as purely fundamental, may also help to realize quantum information tasks. Further investigations will determine whether vacuum entanglement can be used to entangle internal degrees of freedom in a fast way, and to produce spin squeezing, which would be of practical interest in, for example, atomic clocks [19].

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