

Quantum Glassiness in Strongly Correlated Clean Systems: An Example of Topological Overprotection

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(Received 4 April 2004; published 31 January 2005)

This Letter presents solvable examples of quantum many-body Hamiltonians of systems that are unable to reach their ground states as the environment temperature is lowered to absolute zero. These examples, three-dimensional generalizations of quantum Hamiltonians proposed for topological quantum computing, (1) have no quenched disorder, (2) have solely local interactions, (3) have an exactly solvable spectrum, (4) have topologically ordered ground states, and (5) have slow dynamical relaxation rates akin to those of strong structural glasses.

DOI: 10.1103/PhysRevLett.94.040402

PACS numbers: 05.30.-d, 03.67.Lx, 61.43.Fs

Describing matter at near absolute zero temperature requires understanding a system's quantum ground state and the low energy excitations around it, the quasiparticles, which are thermally populated by the system's contact to a heat bath. However, this paradigm breaks down if thermal equilibration is obstructed. While such nonequilibrium behavior may be expected in disordered and frustrated quantum systems (for instance, quantum spin glasses [1], long-range Josephson junction arrays in a frustrating magnetic field [2], or self-generated mean-field glasses [3]), it is not obvious that it may exist in clean systems with only local interactions and without a complicated distribution of energy levels. In this Letter I present solvable examples, three-dimensional generalizations of Hamiltonians proposed for topological quantum computing, that have solely local interactions, no quenched disorder, and relaxation rates akin to those of strong structural glasses. Therefore, in these systems the topologically ordered ground states are not reached when the temperature is reduced to absolute zero.

Topological order and quantum number fractionalization are some of the most remarkable properties of systems of strongly interacting particles. Some phases of matter, in contrast to common examples like crystals and magnets, are not characterized by a local order parameter and broken symmetries. Instead, as shown by Wen [4,5], some quantum phases are characterized by their topological order, such as the degeneracy of the ground state when the system is defined on a torus or other surface of higher genus. These topological degeneracies cannot be lifted by *any* local perturbation. Topological order and quantum number fractionalization are intimately related, and much effort has recently been directed at these exotic properties, for they may play a role in the mechanism for high-temperature superconductivity [6–8]. Also, the robustness of a topological degeneracy to local noise due to an environment is at the core of the idea behind topological quantum computation, as proposed by Kitaev [9]. Interestingly enough, strong correlations that can lead to these exotic quantum spectral properties can in some instances also impose

kinetic constraints, similar to those studied in the context of classical glass formers [10–17].

The possibility of glassiness in pure strongly correlated quantum systems with solely local interactions is demonstrated by studying the following exactly solvable example. A model displaying strong glassiness is constructed on a three-dimensional (3D) face-centered cubic (fcc) Bravais lattice, spanned by the primitive vectors $\mathbf{a}_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $\mathbf{a}_2 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $\mathbf{a}_3 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. Each site can be indexed by $i, j, k \in \mathbb{Z}$, denoted by a superindex $I \equiv (i, j, k)$. At every lattice site I one defines quantum spin $S = 1/2$ operators σ_I^x , σ_I^y , and σ_I^z .

The fcc lattice can house sets of *octahedra*: the simplest one to visualize is the one assembled from the centers of the six faces of a cubic cell, and is shown in Fig. 1(a). In addition to this simple set, there are three more sets of octahedra that can be assembled from sites both on faces

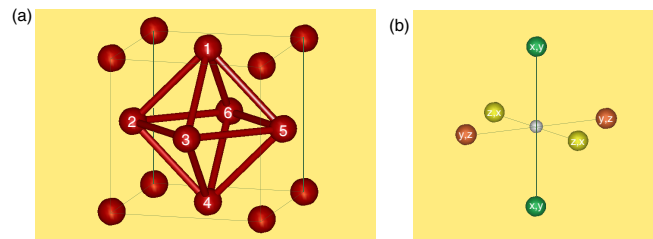


FIG. 1 (color online). (a) Cubic cell of an fcc lattice. The centers of the six faces form an octahedron, with its sites labeled from 1 (topmost) to 6. In addition to the set of octahedra formed by the face-centered sites, there are three more sets of octahedra that can be assembled from sites both on faces and on corners of the cubic cells, totaling four such sets. Six-spin operators are defined on these octahedra using the $\sigma^{x,y,z}$ components of spin on each vertex as described in the text. (b) Centers of six octahedra cells that share a spin, which resides at the site I shown at the center. The x, y, z labels sitting at the centers of the octahedra show which spin operators $\sigma_I^{x,y,z}$ flip their O_I eigenvalue. Acting with any of the operators, $\sigma_I^{x,y,z}$ always flip the eigenvalues O_I of exactly four octahedra.

and on corners of the cubic cells, totaling four such sets, which we label by $A, B, C,$ and D .

It is simple to see that the total number of octahedra equals the number of spins: each lattice site I is the topmost vertex of a single octahedron. Define P_I as the set of six lattice points forming the octahedron with site I at its top. The six vertices are indexed by $J_n(I)$, for $n = 1, \dots, 6$, with one of the vertices $J_1(I) = I$. The six labels are assigned in such a way that the pairs $\{J_1, J_4\}, \{J_2, J_5\}, \{J_3, J_6\}$ are diagonally opposite sites from one another, and this number labeling is illustrated for a single octahedron in Fig. 1(a). From the one-to-one relation between a site I and an octahedron P_I , we can also partition the lattice sites into four sublattices $A, B, C,$ and D .

Now define the operators \mathcal{O}_I as

$$\mathcal{O}_I = \sigma_{J_1(I)}^z \sigma_{J_2(I)}^x \sigma_{J_3(I)}^y \sigma_{J_4(I)}^z \sigma_{J_5(I)}^x \sigma_{J_6(I)}^y. \quad (1)$$

This construction generalizes to 3D the plaquette interactions defined for planar 2D lattices by Kitaev [9] and Wen [18]. These operators commute, $[\mathcal{O}_I, \mathcal{O}_{I'}] = 0$ for all pairs I, I' . It is simple to see how: two octahedra P_I and $P_{I'}$ can either share 0, 1, or, at most, 2 spins. If they share 0 spins, they trivially commute. If they share 1 spin, the component ($x, y,$ or z) of σ for that shared spin coincides for both \mathcal{O}_I and $\mathcal{O}_{I'}$ (the two octahedra touch along one of their diagonals). If they share 2 spins, the components of σ used in the definition of \mathcal{O}_I and $\mathcal{O}_{I'}$ are different for both spins, there is a minus sign from commuting the spin operators from each of the shared spins, and the two minus signs cancel each other.

Consider the Hamiltonian

$$\hat{H} = -\frac{h}{2} \sum_I \mathcal{O}_I. \quad (2)$$

Because the \mathcal{O}_I all commute, the eigenvalues of the Hamiltonian can be labeled by the list of eigenvalues $\{O_I\}$ of all the \mathcal{O}_I . Notice that $\mathcal{O}_I^2 = \mathbb{1}$, and so each $O_I = \pm 1$. In particular, the ground state corresponds to $O_I = 1$ for all I .

Because the number of spins equals the number N of sites and of octahedra, one may naively expect that the list $\{O_I = \pm 1\}$ exhausts the 2^N states in the Hilbert space. However, there are constraints that \mathcal{O}_I satisfy when the system is subject to periodic boundary conditions (compactified). One can show that

$$\prod_{I \in A} \mathcal{O}_I = \prod_{I \in B} \mathcal{O}_I = \prod_{I \in C} \mathcal{O}_I = \prod_{I \in D} \mathcal{O}_I = 1. \quad (3)$$

There are four constraints; therefore, there are only 2^{N-4} independent $\{O_I = \pm 1\}$. This implies, in particular, that there is a ground state degeneracy of $2^4 = 16$. The ground state degeneracy is *not* associated with a symmetry; in particular, it is easy to show that $\langle \sigma_I^{x,y,z} \rangle = 0$. This is a topological degeneracy, and the eigenvalues $T_a = \pm 1$

($a = 1, 2, 3, 4$) of a set of four nonlocal (topological) operators \mathcal{T}_a are needed to distinguish between the 16 degenerate ground states.

The operators \mathcal{T}_a can be constructed as follows. Let $\mathcal{P}_l = \{I | j+k=l\}$ be a set of points along a horizontal plane. Notice that each such plane contains sites in only two of the four sublattices A, B, C, D . For example, $\mathcal{P}_1 \subset A \cup B$ and $\mathcal{P}_2 \subset C \cup D$. Define $\mathcal{T}_1 = \prod_{I \in \mathcal{P}_1 \cap A} \sigma_I^z$, $\mathcal{T}_2 = \prod_{I \in \mathcal{P}_1 \cap B} \sigma_I^z$, $\mathcal{T}_3 = \prod_{I \in \mathcal{P}_2 \cap C} \sigma_I^z$, $\mathcal{T}_4 = \prod_{I \in \mathcal{P}_2 \cap D} \sigma_I^z$. It is simple to check that $[\mathcal{T}_a, \mathcal{O}_I] = 0$ for all a and I , and the \mathcal{T}_a trivially commute among themselves. Hence the four eigenvalues $T_{1,2,3,4} = \pm 1$ of $\mathcal{T}_{1,2,3,4}$ can distinguish the 16 degenerate ground states.

The spectrum of the Hamiltonian Eq. (2) is that of a trivial set of $N - 4$ free spins, determined by the list of eigenvalues $\{O_I = \pm 1\}$ of all the \mathcal{O}_I , subject to the condition Eq. (3): $E_{\{O_I\}} = -\frac{h}{2} \sum_I O_I$. Excitations above the ground state ($O_I = 1$ for all I) are “defects” where $O_I = -1$ at certain sites I . The equilibrium partition function is given by $Z = 16 \sum_{\{O_I = \pm 1\}} e^{(1/2)\beta h \sum_I O_I}$. At thermal equilibrium at temperature T , the thermal average $\langle O_I \rangle = \tanh \frac{h}{2T}$, and the concentration or density of $O_I = -1$ defects is $c = \frac{1}{2}(1 - \tanh \frac{h}{2T})$. Notice that we have encountered an analogous situation to that in the classical spin facilitated models [10], in particular, the plaquette models displaying glassy dynamics [14–17]: the thermodynamics is trivial in terms of noninteracting defect variables. To study the approach to such an equilibrium state, the coupling of the system to a bath of quantum oscillators must be introduced.

The Hamiltonian of the system plus bath of oscillators can be formulated as [19,20]

$$\hat{\mathcal{H}} = \hat{H} + \hat{H}_{\text{bath}} + \hat{H}_{\text{spin/bath}},$$

where \hat{H} is defined in Eq. (2), the bath H_{bath} contains a family of harmonic oscillators $\mathbf{a}_{\lambda,I}, \mathbf{a}_{\lambda,I}^\dagger$ for each site, and

$$H_{\text{spin/bath}} = \sum_{I,\alpha} g_\alpha \sigma_I^\alpha \sum_\lambda (a_{\lambda,I}^\alpha + a_{\lambda,I}^{\alpha\dagger}), \quad (4)$$

where the g_α are the generic coupling constants for each of the three components ($\alpha = 1, 2, 3$ or x, y, z) of the spins.

Although the spectrum of \hat{H} in Eq. (2) is the same as that of free spins in a uniform magnetic field h , the variables O_I for different octahedra P_I *cannot* be independently changed, as opposed to spin variables in a free spin model in a field h . The bath couples to the *physical* degrees of freedom, the spins σ_I . Acting on a site $I \in P_I$ with one of the operators $\sigma_I^x, \sigma_I^y,$ or σ_I^z flips or not the eigenvalue O_I depending on whether $\sigma_I^{x,y,z} \mathcal{O}_I = \mp \mathcal{O}_I \sigma_I^{x,y,z}$, respectively. However, the spin σ_I is shared by six neighboring octahedra, and thus one cannot change the eigenvalue of O_I without changing the eigenvalues $O_{I'}$ of some of the neighbors by the action of the *local* spin operator.

If integrated out, the bath degrees of freedom lead to a nonlocal in time action and to dissipation effects. Instead of working with the dissipative action, let us follow the time evolution of the system plus bath, and look at the possible evolution pathways of the quantum mechanical amplitudes of the system plus bath degrees of freedom. (Yet another alternative is to work within the von Neumann density matrix formalism [21], and follow the time evolution of the matrix elements.) After evolution for time t from some initial state, the system is in a quantum mechanical superposition

$$|\Psi(t)\rangle = \sum_{\{T_a, O_I = \pm 1\}} \Gamma_{\{T_a, O_I\}}(t) |T_a, O_I\rangle \otimes |Y_{\{T_a, O_I\}}(t)\rangle, \quad (5)$$

where $|Y_{\{T_a, O_I\}}(t)\rangle$ is some state in the bath Hilbert space with norm one. The fact that the bath degrees of freedom couple to single quantum spins σ_I [as in Eq. (4)] enters the problem through the permitted channels for dynamically transferring amplitudes among the $\Gamma_{\{T_a, O_I\}}$.

The processes that redistribute or transfer amplitude among the $\Gamma_{\{T_a, O_I\}}$ correspond to different orders in perturbation theory on the g_α system-bath coupling. There is also a thermal probability factor coming from the bath that depends on the difference between the initial and the final energies $E_{\{O_I\}} = -\frac{h}{2} \sum_I O_I$ of the system. The simplest class of paths is a *sequential* passage over states connected through first order in g_α processes; this is a “semiclassical” type trajectory.

Through the action of a local $\sigma_I^{x,y,z}$ operator, exactly four of the six octahedra operators O_I sharing spin I are flipped. The reason is that the six octahedra operators can be divided into three groups of two octahedra having in their definitions Eq. (1), respectively, the x , y , and z components of the spin operator at the shared site. This is illustrated in Fig. 1(b). Acting with any of the three components of the spin operator $\sigma_I^{x,y,z}$ on this shared site will flip exactly four out of six defect variables O_I . Hence, a single defect cannot be annihilated in this process. Defects disappear from the system only through recombination. This multidefect type dynamics makes it difficult for the system to relax to equilibrium, exactly as in the kinetically constrained classical models [14–17]. For example, if the temperature is lowered, in order to decrease the defect density, either four defects must come together and annihilate ($4 \rightarrow 0$ decay) or three defects become one ($3 \rightarrow 1$ decay). Moreover, single defects cannot simply diffuse through the system; that would require flipping only two neighboring octahedra, but instead four are always flipped. To move, an isolated single defect must first decay into three defects ($1 \rightarrow 3$ production) because of the multidefect dynamics, then a pair can diffuse freely ($2 \rightarrow 2$) and recombine with another defect through a $3 \rightarrow 1$ decay process. Because of the initial $1 \rightarrow 3$ production process, there is an energy barrier of $2h$ to be overcome. This activation barrier leads to recombination or equilibration times

$$t_{\text{seq}} \sim \tau_0 \exp(2h/T)$$

that grow in an Arrhenius fashion as temperature is lowered (τ_0 is a microscopic time scale).

What about quantum tunneling? Amplitude can be transferred from some initial to some final state via virtual processes, in which the number of defects is larger in the intermediate (virtual) steps. Virtual processes of n th order involve a product of n spin operators, $\mathcal{F} \equiv \prod_{s=1}^n \sigma_{I_s}^{\alpha_s}$. For a single defect to disperse through quantum tunneling, an \mathcal{F} operator that flips only two octahedra is needed. However, one can show that any \mathcal{F} will flip *at least four* octahedra (as opposed to Kitaev’s and Wen’s models, in which two defects stand at the end points of “strings,” here four or more defects lie at the corners of “membranes”). Although defects cannot disperse, tunneling still contributes to defect annihilation and to defect pair motion. In perturbation theory, a process in which a defect pair separated by a distance ξ can hop by a lattice spacing has an amplitude of order $(g/h)^\xi$ (notice the energy denominator h) and defect annihilation has an amplitude $(g/h)^{\xi^2}$. Hence virtual processes are suppressed exponentially in ξ , and if the system were to equilibrate at temperature T , where the typical defect separation is $\xi = c^{-1/3} \sim e^{h/3T}$, the characteristic time scale

$$t_{\text{tun}} \sim \tau_0 \exp[\ln(h/g)e^{h/3T}]$$

would grow extremely quickly as the temperature is lowered. What we learn from this simple estimate is that quantum tunneling is less effective than classical sequential processes in thermalizing the system. This is counterintuitive to the notion that at low temperatures quantum tunneling under energy barriers remains an open process while classical mechanisms are suppressed due to high thermal activation costs. The reason for the particular quantum freezing in this system is simple: as the distance between defects increases at lower temperatures, the barrier *widths* increase, which debilitates tunneling. In passing, we note that in a finite system of size L , one must replace ξ by L in the estimation of the recombination or equilibration times, $t_{\text{tun}} \sim \tau_0 \exp[\ln(h/g)L]$; this time scale is also of the order of that for tunneling between two topological ground states in a finite system of size L [9].

Because t_{seq} and t_{tun} grow rapidly as the temperature lowers, the system will fall out of equilibrium at low temperatures, and physical correlation functions will not be those simply computed in the framework of equilibrium quantum statistical mechanics. The simplest correlation function that illustrates this point is the one-point function related to the time-dependent spatial concentration of defects $\rho_I(t) = \frac{1}{2}[1 - \langle O_I(t) \rangle]$; let us find how it approaches, as a function of time, the asymptotic equilibrium value $c_f = \frac{1}{2}(1 - \tanh \frac{h}{2T_f})$ when the temperature is, say, reduced by half from T_i to $T_f = T_i/2$. The mechanism for equi-

bration is diffusion-annihilation of defects. We have argued that defects are always flipped in quadruplets, and single defects cannot freely diffuse without generating more defects. Defect pairs, however, are free to diffuse quickly, so we can reduce the problem to an effective reaction diffusion [22] of the $A + A \rightleftharpoons 0$ type for the single defects facilitated by the pair motion. The quantum average over the state Eq. (5) can restore translational invariance of the density $\rho_I(t)$, so spatially homogeneous initial densities remain homogeneous under time evolution, $\rho_I(t) = c(t)$; hence, the dynamics for this problem is controlled by the simple rate equation $\dot{c}(t) = -k(c^2 - c_f^2)$, with the kinetic rate coefficient k directly proportional to the defect diffusion constant at temperature T_f , from which it follows that $k \propto 1/t_{\text{seq}}$. At long times $c(t) - c_f \propto c_f \exp(-2c_f kt)$, from which we extract the time constant for the relaxation of the one-point correlations to be $\tau_{1\text{pt}} = t_{\text{seq}}/2c_f$. Notice that the relaxation time $\tau_{1\text{pt}}$ for the one-point correlation is longer than t_{seq} because the annihilation rate is reduced for low densities of defects. This enhancement must be cut off when the density of defects is of order $1/L^3$, in which case $\tau_{1\text{pt}} \sim L^3 t_{\text{seq}}$. The relaxation time is just polynomial in the system size L , so if t_{seq} saturated to a constant value as the temperature is lowered, the system would not behave as a glass. It is the Arrhenius form of t_{seq} that causes the glassy behavior.

The fact that the system presented above is exactly solvable helps one to understand the origin of its glassy behavior, but it is *not* a necessary ingredient. To illustrate this point, one can simply add a perturbation $\delta\hat{H} = \sum_{I,\alpha} \Delta_\alpha \sigma_I^\alpha$. For Δ_α/h less than order unity, this interaction can be analyzed in perturbation theory similar to the arguments above for the ratios g_α/h . The perturbation will give the defects some mobility, but that becomes exponentially small as the defects grow apart. Indeed, these arguments can be generalized for any local perturbation written in terms of the physical spins σ_I , as long as the coupling constants are small compared to the gap h .

To summarize, the essence of why this quantum system is glassy is the following. The thermodynamics is best described by working in the basis of eigenstates or defects $\{|\mathcal{T}_a, \mathcal{O}_I\rangle\}$; however, upon acting on these states with physical spin operators $\sigma_I^{x,y,z}$, single defects can neither be annihilated nor simply be moved around (diffused). The lack of defect diffusion in these glassy systems is protected by the fact that any physical spin operators must flip quadruplets, not pairs, of defect variables. The system can relax only by multidefect real processes that are thermally suppressed or else by virtual processes that involve quantum tunneling of increasingly large objects as the defect density is reduced at low temperatures.

Many elastic, thermal, electronic, and magnetic properties of classical glassy material systems are consequences of these materials being out of equilibrium. Such properties

can be tailored according to preparation schemes, for example, by controlling cooling rates. In contrast, because of the difficulties in studying real-time dynamics of strongly interacting quantum systems coupled to a thermal bath, very little is currently known about properties of quantum matter that can be engineered by keeping systems out of equilibrium. In a broader scope, the main result of this work is that it presents a concrete example of a solvable toy model which shows without arbitrary or questionable approximations that a pure quantum system with only local interactions can, indeed, stay out of equilibrium. This result supports the possibility that there may be material properties due to nonequilibrium glassy behavior in quantum matter. It also suggests a new design constraint for topological quantum computing: that the ground state degeneracy is protected while the system is still able to reach the ground states.

The author thanks G. Biroli, C. Castellano, A. Castro-Neto, L. Cugliandolo, M. P. Kennett, C. Nayak, E. Novais, P. Pujol, D. Reichman, A. Sandvik, and X.-G. Wen for illuminating discussions. This work is supported in part by the NSF Grant No. DMR-0305482.

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