Topological Susceptibility in SU(3) Gauge Theory

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We compute the topological susceptibility for the SU(3) Yang-Mills theory by employing the expression of the topological charge density operator suggested by Neuberger's fermions. In the continuum limit we find $r_0^4\chi = 0.059(3)$, which corresponds to $\chi = (191 \pm 5\ \mathrm{MeV})^4$ if F_K is used to set the scale. Our result supports the Witten-Veneziano explanation for the large mass of the η'.

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Introduction.—The topological susceptibility in the pure Yang-Mills (YM) gauge theory can be formally defined in Euclidean space-time as

$$\chi = \int d^4x \langle q(x)q(0)\rangle, \tag{1}$$

where the topological charge density $q(x)$ is given by

$$q(x) = -\frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)]. \tag{2}$$

Besides its interest within the pure gauge theory, χ plays a crucial role in the QCD-based explanation of the large mass of the η' meson proposed by Witten and by Veneziano (WV) a long time ago [1,2]. The WV mechanism predicts that at the leading order in $N_\mathrm{f}/N_\mathrm{c}$, where N_f and N_c are the number of flavors and colors, respectively, the contribution due to the anomaly to the mass of the $U_\mathrm{A}(1)$ particle is given by [1–5]

$$\frac{F_\pi^2 m_{\eta'}^2}{2N_\mathrm{f}} = \chi, \tag{3}$$

where F_π is the corresponding pion decay constant (in our conventions $F_\pi = 92$ MeV). Notice that Eq. (3) is expected to be exactly satisfied if the left-hand side is computed in full QCD and the right-hand side in the pure gauge theory, both in the 't Hooft large-N_c limit [6].

The lattice formulation of gauge theories is at present the only approach where nonperturbative computations can be performed with controlled systematic errors. Recent theoretical developments [7–10] (for a recent review, see [11]) led to the discovery of a fermion operator [12–14] that satisfies the Ginsparg-Wilson (GW) relation [15], and therefore preserves an exact chiral symmetry at finite lattice spacing [16]

$$\psi \to \hat{\gamma}_5\psi, \qquad \bar{\psi} \to \bar{\psi}\gamma_5, \tag{4}$$

where $\hat{\gamma}_5 = \gamma_5(1 - \bar{a}D)$, D is the massless Dirac operator, and $\bar{a}$ is proportional to the lattice spacing (see below). The corresponding Jacobian is nontrivial [16], and the chiral anomaly is recovered *à la* Fujikawa [17] with the topological charge density operator defined as [18] (we use the same notation for analogous quantities in the continuum and on the lattice, since they can be clearly distinguished from the context)

$$q(x) = -\frac{\bar{a}}{2}\mathrm{Tr}[\gamma_5 D(x,x)], \tag{5}$$

where the trace runs over spin and color indices. These developments triggered a breakthrough in the understanding of the topological properties of the YM vacuum. They made it possible to find an unambiguous definition of the topological susceptibility with a finite continuum limit [4,19,20], which is independent of the details of the lattice definition [20]. If the charge density suggested by GW fermions $Q \equiv \sum_x q(x) = n_+ - n_-$, with n_+ (n_-) the number of zero modes of D with positive (negative) chirality in a given background, is employed, the suggestive formula

$$\chi = \lim_{\substack{a\to 0 \\ V\to\infty}} \frac{\langle Q^2\rangle}{V} \tag{6}$$

is recovered, where V is the volume. An immediate consequence is an unambiguous derivation of the WV formula [4] which, thanks to new simulation algorithms [21], allows for a nonperturbative investigation of the WV mechanism with controlled systematics.

In the past the topological properties of the pure gauge theory were investigated with fermionic [22,23] and bosonic methods [24–32]. These results, however, are affected by model-dependent systematic errors that are not quantifiable, and their interpretation rests on a weak theoretical ground. Several exploratory computations have already studied the susceptibility employing the GW definition of the topological charge [33–40].

The aim of this work is to achieve a precise and reliable determination of χ in the continuum limit. In order to reach a robust estimate of the error on the extrapolated value, we supplement the most recent and accurate results [39,40] with additional simulations, and we perform a detailed analysis of the various sources of systematic uncertainties. The result for the adimensional scaling quantity computed

on the lattice is $r_0^4\chi = 0.059(3)$, where r_0 is a low-energy reference scale [41]. In physical units, it corresponds to $\chi = (191 \pm 5\ \mathrm{MeV})^4$ if F_K is used to set the scale. Our result supports the WV explanation for the large mass of the η' meson within QCD.

Lattice computation.—The numerical computation is performed by standard Monte Carlo techniques. The ensembles of gauge configurations are generated with the standard Wilson action and periodic boundary conditions, using a combination of heat-bath and over-relaxation updates. More details on the generation of the gauge configurations can be found in Refs. [39,40]. Table I shows the list of simulated lattices, where the bare coupling constant $\beta = 6/g_0^2$, the linear size L/a in each direction, and the number of independent configurations are reported for each lattice.

The topological charge density is defined as in Eq. (5), with D being the massless Neuberger-Dirac operator:

$$D = \frac{1}{\bar{a}}[1 + \gamma_5 \mathrm{sgn}(H)], \tag{7}$$

$$H = \gamma_5(aD_\mathrm{w} - 1 - s), \qquad \bar{a} = \frac{a}{1+s}. \tag{8}$$

Here s is an adjustable parameter in the range $|s| < 1$, and D_w denotes the standard Wilson-Dirac operator (the notational conventions not explained here are as in Ref. [21]). For a given gauge configuration, the topological charge is computed by counting the number of zero modes of D with the algorithm proposed in Ref. [21]. As s is varied, D defines a one-parameter family of fermion discretizations, which correspond to the same continuum theory but with different discretization errors at finite lattice spacing. Our analysis includes data sets computed for $s = 0.4$ and $s = 0.0$. Most of the data were taken from Refs. [39,40] and for $s = 0.0$ and $s = 0.4$, respectively. The number of configurations was increased, where necessary, in order to achieve homogeneous statistical errors of the order of 5% for each data point. Some new lattices were added so as to perform careful studies of the systematic uncertainties which we describe below, before presenting the physical results.

TABLE I. Simulation parameters and results. For lattices A_1–D and E–J, $s = 0.4$ and $s = 0.0$, respectively.

Lattice	β	L/a	r_0/a	L (fm)	N_conf	$\langle Q^2\rangle$	$r_0^4\chi$
A_1	6.0	12	5.368	1.12	2452	1.633(48)	0.0654(22)
A_2	6.1791	16	7.136	1.12	1138	1.589(76)	0.0629(32)
A_3	5.8989	10	4.474	1.12	1460	1.737(72)	0.0696(30)
A_4	6.0938	14	6.263	1.12	1405	1.535(63)	0.0615(27)
B_0	5.8458	12	4.032	1.49	2918	5.61(16)	0.0715(22)
B_1	6.0	16	5.368	1.49	1001	5.58(28)	0.0707(37)
B_2	6.1366	20	6.693	1.49	963	4.81(24)	0.0604(32)
B_3	5.9249	14	4.697	1.49	1284	5.59(24)	0.0708(33)
C_0	5.8784	16	4.301	1.86	1109	15.02(72)	0.0784(39)
C_1	6.0	20	5.368	1.86	931	12.76(95)	0.0662(50)
D	6.0	14	5.368	1.30	1577	3.01(12)	0.0651(27)
E	5.9	12	4.483	1.34	1349	2.79(12)	0.0543(24)
F	5.95	12	4.917	1.22	1291	1.955(79)	0.0551(24)
G	6.0	12	5.368	1.12	3586	1.489(37)	0.0596(18)
H	6.1	16	6.324	1.26	962	2.45(13)	0.0599(33)
J	6.2	18	7.360	1.22	1721	2.114(76)	0.0591(24)

In order to compute its autocorrelation time, we monitor the topological charge determined with the index of D for 500 update cycles (one heat bath and six over-relaxations of all link variables) for the lattice A_1. The autocorrelation time, τ_Q, estimated as in Ref. [32], turns out to be compatible with the one obtained for the same lattice by defining the topological charge with the cooling technique adopted in Ref. [32]. Based on the experience with cooling, where longer Monte Carlo histories can be analyzed, we estimate τ_Q for all our lattices; for each run we separate subsequent measurements by a number of update cycles 1–2 orders of magnitude larger than the estimated τ_Q at the corresponding value of β. Statistical errors are thus computed assuming that the measurements are statistically independent.

Besides the statistical errors, the systematic uncertainties stem from finite-volume effects and from the extrapolation needed to reach the continuum limit.

The pure gauge theory has a mass gap, and therefore the topological susceptibility approaches the infinite-volume limit exponentially fast with L. Since the mass of the lightest glueball is around 1.5 GeV, finite-volume effects are expected to be far below our statistical errors as soon as $L \geq 1$ fm. In order to further verify that no sizable finite-volume effects are present in our data, we simulated four lattices at $\beta = 6.0$ but with different linear sizes $L = 1.12$, 1.30, 1.49, and 1.86 fm. The results obtained for χ are shown in Fig. 1, where no dependence on L is visible, hence confirming that finite-volume effects are below our

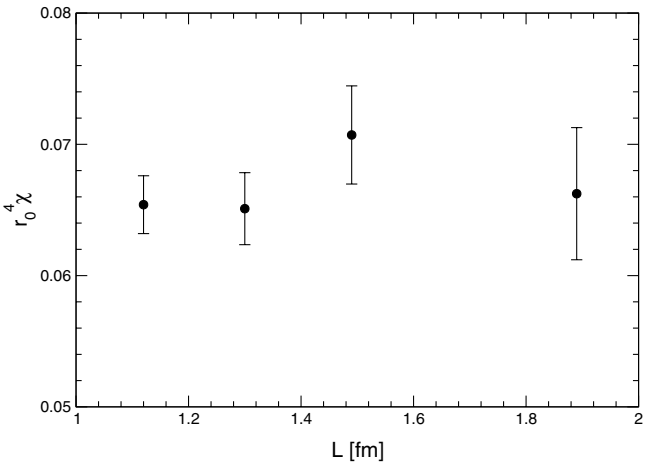

FIG. 1. The topological susceptibility, in units of r_0^{-4}, as a function of the linear lattice size, in fm, at $\beta = 6.0$.

statistical errors. In the large-volume regime the probability distribution of the topological charge is expected to be a Gaussian of the form [40]

$$P_Q = \frac{1}{\sqrt{2\pi\langle Q^2\rangle}} e^{-Q^2/2\langle Q^2\rangle}. \quad (9)$$

We have checked that this formula describes all our data samples very well; for the lattice D, the results are shown in Fig. 2. Much higher statistics are required in order to highlight the deviations from a Gaussian distribution; higher momenta of the topological charge distribution measured on our data are all compatible with zero within large statistical errors.

As pointed out in the introduction, the topological susceptibility defined from the index of the Neuberger operator is not plagued by power divergences and does not require multiplicative renormalization. This is a distinctive feature of this approach, which is at variance with what happens for other definitions used in the past to compute χ. At finite lattice spacing, χ is affected by discretization effects starting at $O(a^2)$, which are not universal, and, in our case, depend on the value of s chosen to define the Neuberger operator. In order to compare results at different lattice spacings, and to extrapolate them to the continuum limit, we adopt r_0 as the reference scale; this choice is motivated by its precise determination in the range of β explored in this work [41]. The values of the adimensional quantity $r_0^4\chi$ that we obtain are reported in Table I. Data, displayed in Fig. 3 as a function of a^2/r_0^2, show sizable $O(a^2)$ effects for both the $s = 0.4$ and $s = 0.0$ samples. For $\beta \leq 6.0$, the difference between the two discretizations is statistically significant. Within our statistical errors, and in the range where our simulations are performed, our results suggest a linear dependence in a^2. For the $s = 0.4$ sample, the value of χ^2 per degree of freedom, $\chi^2_{\rm dof}$, clearly disfavors a constant behavior, while a linear fit of the form

$$r_0^4\chi(s) = c_0 + c_1(s)\left(\frac{a}{r_0}\right)^2 \quad (10)$$

yields a value of $c_0 = 0.056(3)$ with $\chi^2_{\rm dof} \approx 0.79$. The quadratic fit in a^2/r_0^2 yields an extrapolated value compatible with that of the linear one, but with an error 3 times larger, and the coefficient of the quadratic term compatible with zero. For the $s = 0.0$ sample, all three fits give good values of $\chi^2_{\rm dof}$, and for the linear one we obtain $c_0 = 0.064(4)$ with $\chi^2_{\rm dof} \approx 0.68$, which is compatible with the outcome of the same fit for $s = 0.4$. The agreement between the two extrapolations indicates that we reached the scaling regime. This is confirmed by the compatibility of the results in the two data sets for $\beta > 6.0$. A robust estimate of χ in the continuum limit can thus be obtained by performing a combined linear fit of the data. This fit gives a very good value of $\chi^2_{\rm dof}$ when all sets are included and is very stable if some points at larger values of a^2/r_0^2 are removed. In particular, a combined fit of all points with $a^2/r_0^2 < 0.05$ gives $c_0 = 0.059(3)$ with $\chi^2_{\rm dof} \approx 0.73$, and the error is expected to be Gaussian.

Physical results.—From the previous analysis, our best result for the topological susceptibility is the one obtained from a combined fit of the two sets of data with $a^2/r_0^2 < 0.05$:

$$r_0^4\chi = 0.059 \pm 0.003, \quad (11)$$

which is the main result of this work. Since r_0 is not directly accessible to experiments, we express our result in physical units by using the lattice determination of

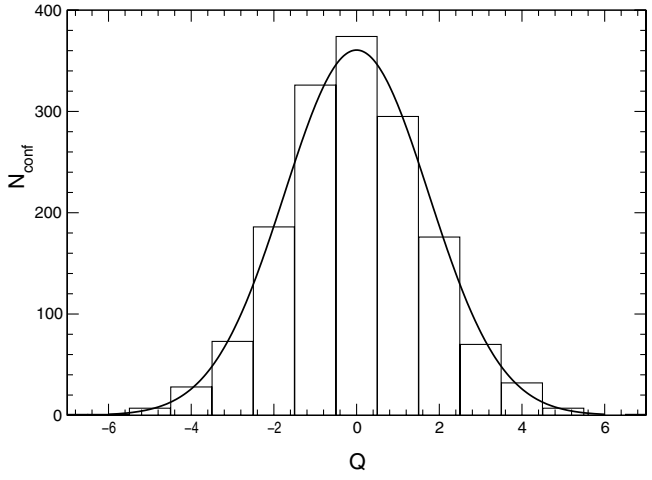

FIG. 2. Histogram for the distribution of the topological charge Q from the lattice D.

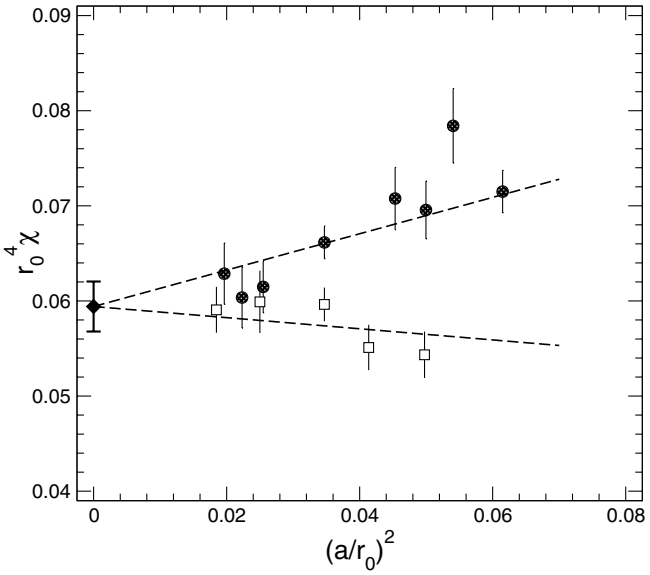

FIG. 3. Continuum extrapolation of the adimensional product $r_0^4\chi$. The $s = 0.0$ and $s = 0.4$ data sets are represented by black circles and white squares, respectively. The dashed lines represent the results of the combined fit described in the text. The filled diamond at $a = 0$ is the extrapolated value in the continuum limit.

$r_0 F_K = 0.4146(94)$ in the pure gauge theory with valence quarks [42] and, taking $F_K = 160(2)$ MeV as an experimental input, we obtain

$$\chi = (191 \pm 5 \text{ MeV})^4, \tag{12}$$

which has to be compared with [2]

$$\frac{F_\pi^2}{6}(m_\eta^2 + m_{\eta'}^2 - 2m_K^2)|_{\text{exp}} \simeq (180 \text{ MeV})^4. \tag{13}$$

Notice that, since Eq. (3) is valid only at the leading order in a N_f/N_c expansion, the ambiguity in the conversion to physical units in the pure gauge theory is of the same order as the neglected terms.

Our result supports the fact that the bulk of the mass of the pseudoscalar singlet meson is generated by the anomaly through the Witten-Veneziano mechanism.

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