Half-Filled Lowest Landau Level on a Thin Torus

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We solve a model that describes an interacting electron gas in the half-filled lowest Landau level on a thin torus, with radius of the order of the magnetic length. The low-energy sector consists of noninteracting, one-dimensional, neutral fermions. The ground state, which is homogeneous, is the Fermi sea obtained by filling the negative energy states, and the excited states are gapless neutral excitations out of this one-dimensional sea. Although the limit considered is extreme, the solution has a striking resemblance to the composite fermion description of the bulk $\nu = 1/2$ state—the ground state is homogeneous and the excitations are neutral and gapless. This suggests a one-dimensional Luttinger liquid description, with possible observable effects in transport experiments, of the bulk state where it develops continuously from the state on a thin torus as the radius increases.

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It was observed by Jiang *et al.* in 1989 that the quantum Hall (QH) system has a metallic behavior at filling fraction $\nu = 1/2$—σ_{xx} is finite and sample dependent as $T \to 0$, whereas σ_{xy} is unquantized and approximately equal to its classical value [1]. Early experiments also showed that there is a large density of low-energy states [2], but no nonlocal transport [3].

The metallic $\nu = 1/2$ state was successfully described by Halperin, Lee, and Read [4], who introduced a mean field theory where the external magnetic field is cancelled by a smeared out statistical field, resulting in composite fermions [5] moving in a zero field—i.e., in a two-dimensional free fermion gas with a Fermi surface. This picture, as well as Jain's general concept of composite fermions, was spectacularly confirmed by surface acoustic wave experiments performed by Willett *et al.* [6], and by ballistic transport [7], which showed that particles move in a reduced effective magnetic field (or, alternatively, have a reduced charge) which approaches zero as $\nu \to 1/2$.

Rezayi and Read [8] proposed a microscopic wave function for the $\nu = 1/2$ state, which agrees very well with exact results for small systems. The theory for the $\nu = 1/2$ state was later further developed by several groups [9], and a description in terms of neutral dipoles was proposed.

In spite of the impressive agreement between theory and experiment there are, in our opinion, important questions regarding the physics in the lowest Landau level that remain to be answered. There is no real understanding of why the strongly correlated electron system in the lowest Landau level, at various filling fractions, becomes a system of weakly interacting composite fermions—no controlled microscopic derivation of the mean field theory at $\nu = 1/2$ or, for that matter, of the composite fermion descriptions at other filling fractions exists.

Here we study the interacting electron gas in the lowest Landau level at $\nu = 1/2$ on a thin torus. We obtain an exact solution for a particular short-range electron-electron interaction that is relevant for a torus with circumference L_1 of the order of the magnetic length. The low-energy sector consists of free neutral one-dimensional fermions. Expressed in terms of the original electrons, these "composite fermions" are nearest neighbor electron-hole pairs, excitons, with a hard-core constraint. This thus provides a dipole picture of the $\nu = 1/2$ state. The ground state is a homogeneous Fermi sea of the neutral fermions, which supports gapless neutral excitations.

The low-energy sector has many features in common with the bulk $\nu = 1/2$ state, such as a homogeneous ground state and gapless neutral excitations, and we conjecture that it develops continuously into the bulk state as $L_1 \to \infty$, rather than being separated from this by a phase transition, see Fig. 1. This suggests a one-dimensional description of the bulk $\nu = 1/2$ state as a Luttinger liquid rather than as a two-dimensional Fermi gas [10].

We start by mapping the problem onto a one-dimensional lattice model. Following Haldane and Rezayi we consider an electron confined to the lowest Landau level on a torus with lengths L_1, L_2 in the x and y directions, respectively [11]. In Landau gauge, $\vec{A} = -By\hat{x}$, the magnetic translation operators become

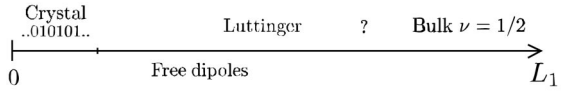

FIG. 1. Phase diagram for $\nu = 1/2$ on a torus, where one circumference, L_1, varies while the other is infinite. For a thin torus, $L_1 \sim 5$ magnetic lengths, the $\nu = 1/2$ system is that of noninteracting neutral one-dimensional fermions (dipoles). The question mark indicates whether the state develops continuously into the bulk $\nu = 1/2$ state or whether there is a phase transition. Based on the similarities of the state at short L_1 and the bulk state, such as a homogeneous ground state and neutral gapless excitations, we conjecture that there is no phase transition. This suggests a one-dimensional description of the bulk $\nu = 1/2$ state as a Luttinger liquid rather than as a free two-dimensional Fermi gas. (For very short L_1 a crystalline state determined by electrostatics alone is obtained.)

$$T_1 = e^{(L_1/N_s)\partial_x}, \qquad T_2 = e^{(L_2/N_s)(\partial_y + ix)}, \tag{1}$$

where N_s is the number of flux quanta through the surface of the torus. (The magnetic length is set to 1, $\ell = \sqrt{\hbar c/eB} = 1$.) These operators commute with the Hamiltonian for the charged particle coupled to $\vec{A}$. Wave functions are required to be periodic up to a phase, $T_\alpha^{N_s}\Psi = e^{i\phi_\alpha}\Psi$, $\alpha = 1, 2$, leading to $L_1 L_2 = 2\pi N_s$ and $T_1 T_2 = e^{2\pi i/N_s} T_2 T_1$. With $\psi_0 = \sum_n e^{inL_2 x} e^{-(y+nL_2)^2/2}$, we obtain the T_1 eigenstates $\psi_m = T_2^m \psi_0$, $T_1 \psi_m = e^{i2\pi m/N_s}\psi_m$, $m = 0, 1, \ldots N_s - 1$. The states ψ_m span the lowest Landau level and are located along the lines $y = -2\pi m/L_1$. Thus we have obtained the mapping onto a one-dimensional lattice model, where m numbers the sites and the lattice constant is $2\pi/L_1$.

Assuming translation invariance, the electron-electron interaction Hamiltonian becomes

$$H_{ee} = \sum_n \sum_{k>m} V_{km} c_{n+m}^\dagger c_{n+k}^\dagger c_{n+m+k} c_n, \tag{2}$$

where $c_m^\dagger$ creates an electron in state ψ_m and $V_{km} = V_{k,-m} \geq 0$. To understand the physics of the interaction it is useful to divide H_{ee} into two parts: V_{p0}, the electrostatic repulsion between two electrons separated p lattice constants, and $V_{m+p,m}$, the amplitude for two particles separated a distance p to hop symmetrically to a separation $p + 2m$ and vice versa. For a short-range real space electron-electron interaction of the form $V(\vec{r}) = \nabla^2\delta(\vec{r})$ one finds $V_{km} = (k^2 - m^2)e^{-2\pi^2(k^2+m^2)/L_1^2}$ [12]. When the torus becomes thin, i.e., when L_1 decreases, the distance $2\pi/L_1$ between the single particle states increases, hence fewer terms in (2) contribute. For the $\nabla^2\delta$ interaction one finds that the range of the interaction is of the order of six magnetic lengths.

We consider the electron gas at filling fraction $\nu = 1/k$, where k is an integer, and assume that the number of electrons νN_s is an integer. The many particle states can be chosen as T_1 eigenstates with momentum $\kappa \bmod(N_s)$ (in units of $2\pi/N_s$). T_2 translates the system in the y direction and changes κ by νN_s—i.e., by the number of particles. Since T_2 commutes with the Hamiltonian, all energy eigenstates are k-fold degenerate [11]. (T_2^k preserves $\kappa \bmod(N_s)$ and hence can be diagonalized along with T_1.)

At a fixed filling fraction, the electrostatic repulsion strives to keep the particles apart, whereas the hopping terms favor maximally hoppable states. To find the low-energy states is in general a very complicated problem. However, at $\nu = 1/2$, the short-range electrostatic and hopping terms cooperate, leading to a simple low-energy sector for the thin torus.

We truncate the interaction in (2) and keep only the two shortest range electrostatic terms and the shortest range hopping term

$$H = \sum_n [V_{10}\hat{n}_n\hat{n}_{n+1} + V_{20}\hat{n}_n\hat{n}_{n+2} - V_{21}(c_n^\dagger c_{n+1} c_{n+2} c_{n+3}^\dagger + \text{H.c.})], \tag{3}$$

where $\hat{n}_k = c_k^\dagger c_k$. This provides a good approximation of the interaction on a thin torus as discussed below [13].

Before giving the details of our analysis we outline the logic of the identification of the low-energy sector of the Hamiltonian (3) at $\nu = 1/2$. The crucial part in (3) is the hopping term V_{21}. We define a subspace $\mathcal{H}'$ of the full Hilbert space by requiring each pair of sites $(2p-1, 2p)$ to have charge one. Acting with T_2 gives an equivalent grouping of the sites $(2p, 2p+1)$ instead—and a corresponding subspace $\mathcal{H}'_T$ [14]. $\mathcal{H}'$ (and $\mathcal{H}'_\mathcal{T}$) is the low-energy sector under fairly general conditions since it contains the maximally hoppable state $|100110011001\ldots\rangle$—which turns out to be the seed for the ground state and is also the lowest energy state for the V_{20} term—and it has a low electrostatic energy by construction. H preserves the subspace $\mathcal{H}'$ and the hopping term can be exactly diagonalized in this space giving noninteracting neutral fermions. The ground state is the one-dimensional Fermi line obtained by filling the negative energy states, and the excitations are gapless excitations out of this Fermi sea. The electrostatic terms in (3) are less crucial. At $V_{10} = 2V_{20}$ all states in $\mathcal{H}'$ have the lowest possible electrostatic energy and we argue perturbatively that $\mathcal{H}'$ is the low-energy sector. However, we expect this to be true under more general conditions.

We now present our analysis for the truncated Hamiltonian (3) at $\nu = 1/2$ and $V_{10} = 2V_{20} \equiv 2\alpha$. The electrostatic part, $H|_{V_{21}=0}$, then has the eigenstates $|n_1 n_2 \ldots n_{N_s}\rangle$, where $n_i = 0, 1$ and $|1\rangle = c^\dagger|0\rangle$, with energies

$$E_0 = \alpha\left(\frac{N_s}{2} + n_{111} + n_{000}\right). \tag{4}$$

Here, $n_{111}(n_{000})$ is the number of 3-strings, i.e., strings consisting of three nearby electrons (holes) in $n_1 n_2 \ldots n_{N_s}$ (a string of length $k \geq 3$ is counted as $k-2$ strings and periodic boundary conditions are assumed). Thus there is a degenerate ground state manifold $\mathcal{H}_0$ consisting of all states where at most two electrons or two holes are next to each other. Note that $\mathcal{H}' \subset \mathcal{H}_0$. The excitations are 3-strings of either electrons or holes and each 3-string has energy α. The statement about $\mathcal{H}_0$ follows by induction if the states without 3-strings for N_s sites are constructed by inserting an electron and a hole in $N_s - 2$ states without 3-strings. The energy of a 3-string follows by considering the change in energy of one or several 3-strings when moving one constituent.

To diagonalize H in $\mathcal{H}'$, we proceed as follows. There are two possible states for a pair of sites in $\mathcal{H}'$: $|\downarrow\rangle \equiv |01\rangle$ and $|\uparrow\rangle \equiv |10\rangle$, and it is natural to introduce the spin raising operator $s_p^+ = c_{2p-1}^\dagger c_{2p}$, $|\uparrow\rangle = s^+|\downarrow\rangle$. On states

in $\mathcal{H}'$, $s^+, s^- = (s^+)^\dagger$ describe hard-core bosons—they commute on different sites but anticommute on the same site and H is the nearest neighbor spin-1/2 XY chain. Expressing the bosons in terms of fermions d using the Jordan-Wigner transformation, $s_p^- = K_p d_p$, where $K_p = e^{i\pi\sum_{j=1}^{p-1} d_j^\dagger d_j}$, the Hamiltonian (3) is simply that of free fermions, $H = \frac{\alpha N_s}{2} + V_{21}[\sum_{p=1}^{N_s/2-1} d_{p+1}^\dagger d_p + d_1^\dagger K_{N_s/2} d_{N_s/2} + \text{H.c.}]$, when restricted to $\mathcal{H}'$ [15]. Thus, after a Fourier transformation, the ground state is obtained by filling all the negative energy states. This state has energy $E = \frac{\alpha}{2} - \frac{V_{21}}{\pi}$ per site (if $N_s \to \infty$) and supports neutral gapless excitations. One readily finds that $\langle c_m^\dagger c_n \rangle = \frac{1}{2}\delta_{mn}$, and hence the state is homogeneous.

This solves the problem in $\mathcal{H}'$ and, by action of T_2, in $\mathcal{H}'_{\mathcal{T}}$. It remains to consider the states in the ground state manifold $\mathcal{H}_0$ that are in neither of these subspaces. We will now argue that these are separated from the ground state by a gap of order V_{21} generated by the hopping term in H. Intuitively, this makes sense since $\mathcal{H}'$ contains the maximally hoppable state $|01100110011\ldots110\rangle$. Note that whereas $\mathcal{H}'$ is invariant under H, other states in $\mathcal{H}_0$ may mix with states not in the ground state manifold. Our procedure will be to simply diagonalize H in the ground state manifold $\mathcal{H}_0$.

To describe a general state in $\mathcal{H}_0$, we introduce the notation $|a\rangle \equiv |00\rangle$ and $|b\rangle \equiv |11\rangle$, along with $|\downarrow\rangle$ and $|\uparrow\rangle$, for the states on sites $(2p-1, 2p)$. H contains the hopping terms: $\uparrow\downarrow \leftrightarrow \downarrow\uparrow$, $\downarrow a \uparrow \leftrightarrow aba$, $\uparrow b \downarrow \leftrightarrow bab$, $\uparrow ba \leftrightarrow ba \uparrow$, and $\downarrow ab \leftrightarrow ab \downarrow$—all with strength V_{21}. A general state is uniquely described by a string of $\uparrow, \downarrow, a$ and b and can be characterized by the number d of alternating $\mathcal{H}'$ and $\mathcal{H}'_{\mathcal{T}}$ domains of which it is built up. Any pair $\uparrow\downarrow$ or $\downarrow\uparrow$ belongs to $\mathcal{H}'$ and any a or b belongs to $\mathcal{H}'_{\mathcal{T}}$—a polarized string $\uparrow\uparrow\uparrow\uparrow\ldots$ or $\downarrow\downarrow\downarrow\downarrow\ldots$ can, however, belong to either domain. This implies that there is a domain wall in between $\uparrow\downarrow$ (or $\downarrow\uparrow$) and a (or b)—counting the number of domain walls gives d for a general state.

A state is a $\nu = 1/2$ state if it has an equal number of a's and b's and belongs to $\mathcal{H}_0$ if it does *not* contain any of the nearest neighbor combinations (aa), (bb), $(a\downarrow)$, $(b\uparrow)$, $(\uparrow a)$, or $(\downarrow b)$. It is straightforward to show that d is preserved by H. The states in $\mathcal{H}'$ and $\mathcal{H}'_{\mathcal{T}}$ are the $d = 1$ states. In the $d = 2$ sector, we consider first the states with one a and one b next to each other, ab or ba, in a string of spins. These $d = 2$ states are mapped into each other under H. To be in $\mathcal{H}_0$, ab must enter as $X \equiv \downarrow ab \downarrow$. Under H, this hops just like $\uparrow : X\downarrow \leftrightarrow \downarrow X$ with matrix element V_{21}. Thus the problem is equivalent to the $\mathcal{H}'$ problem with $N_s - 6$ sites and one finds that these states are separated from the ground state in $\mathcal{H}'$ by a gap of order V_{21}. By considering how hoppable the states are, we expect the states just considered to be the lowest energy states in the $d = 2$ sector. We have verified this by exact diagonalization in $\mathcal{H}_0$ of up to $N_s = 18$ sites. The $d \geq 3$ states contain more domain walls and it is easy to see that the hopping becomes more restricted—thus we expect them to have higher energy. We have verified this numerically for $N_s \leq 18$. Thus we conclude that $\mathcal{H}'$ and $\mathcal{H}'_{\mathcal{T}}$ give the low-energy sector of the theory—the remaining states in $\mathcal{H}_0$ are separated from the ground state by a gap of order V_{21}.

We now consider the stability of the solution we have obtained for H in (3) when $V_{10} = 2V_{20}$ and investigate whether it describes the $\nu = 1/2$ state on a thin torus for a range of L_1. We first note that for the real space short-range interaction $V(\vec{r}) = \nabla^2\delta(\vec{r})$, $V_{10} = 2V_{20} = 2\alpha$ corresponds to $L_1 = 2\pi/\sqrt{2\ln 2} = 5.3$. The hopping term included in (3) is then $V_{21} = \frac{3}{8}\alpha$, whereas the leading ignored terms are small: $V_{30} = \frac{9}{128}\alpha$ and $V_{31} = \frac{1}{32}\alpha$. This is close to the solvable point.

We have performed density matrix renormalization group (DMRG) [16] studies on a thin cylinder with the Hamiltonian (2) and $V(\vec{r}) = \nabla^2\delta(\vec{r})$ including interactions that extend over up to six lattice constants [17]. We find a ground state that is homogeneous to very high accuracy and strong indications of gapless excitations in the region around $L_1 = 2\pi/\sqrt{2\ln 2}$ that we have investigated ($4 \lesssim L_1 \lesssim 8$). (When L_1 is even smaller, the ground state is a crystal $|\uparrow\uparrow\uparrow\uparrow\uparrow\ldots\uparrow\rangle$—the lowest energy state for the shortest range electrostatic term V_{10}.)

The low-energy sector at the solvable point (H in (3) with $V_{10} = 2V_{20}$) is contained in the spin-1/2 Hilbert space $\mathcal{H}'$. The states not in $\mathcal{H}'$ are separated from the low-energy states by a gap. Small perturbations of the Hamiltonian around the solvable point lead, in perturbation theory, to an effective spin-1/2 Hamiltonian in $\mathcal{H}'$. The generated terms are spin operators of quadratic and higher order. They all have small coefficients since there is a gap to states not in $\mathcal{H}'$ (the matrix elements for transitions to states in $\mathcal{H}'_{\mathcal{T}}$ vanish). All terms are irrelevant in the sense of the renormalization group, except for $s_i^z s_{i+n}^z$ which makes the noninteracting fermion theory develop into a Luttinger liquid with interaction parameter $K \neq 1$ (at the solvable point $K = 1$), see, e.g., [18].

To obtain the general effective Hamiltonian explicitly is nontrivial. However, the electrostatic terms, V_{m0}, preserve $\mathcal{H}'$ and simply become $\sum_{i,n}[(2V_{2n,0} - V_{2n-1,0} - V_{2n+1,0})s_i^z s_{i+n}^z]$. The hopping terms, V_{mn}, will, in general, contribute in second order perturbation theory.

Based on the renormalization group argument, and supported by the DMRG calculations, we conclude that the $\nu = 1/2$ system on a thin torus is a Luttinger liquid for a finite range of L_1 and that the generation of $s_i^z s_{i+n}^z$ terms indicates that the interaction parameter that determines the decay of correlation functions is shifted from its value at the solvable point.

When L_1 increases further, there is either a phase transition or the state develops continuously into the bulk $\nu = 1/2$ state. We conjecture that the latter is the case. The main support for this comes from the striking similarities

of the low-energy sector on the thin torus and the composite fermion description of the bulk state, most notably the homogeneous ground state and the gapless neutral excitations. Furthermore, we note that the reduction of the Hilbert space to $\mathcal{H}'$ by itself implies that the charge on average is homogeneous—this, or some suitable generalization thereof, is likely to be a good approximation also when L_1 increases and longer range interactions come into play.

Further support for our conjecture comes from considering the Laughlin filling fraction $\nu = 1/3$ [19]. Rezayi and Haldane have shown that the Laughlin state is the $\nu = 1/3$ ground state also on a thin cylinder and that it develops continuously from a charge density wave state into the homogeneous Laughlin state as $L_1 \to \infty$ [20]. Our DMRG calculations agree with this—we find, using (2), for a range of L_1 a charge density wave state in quantitative agreement with that of Rezayi and Haldane. Thus the $\nu = 1/3$ ground state of the short-range Hamiltonian develops continuously into the homogeneous Laughlin state as $L_1 \to \infty$. This lends some support for our conjecture that the $\nu = 1/2$ state also develops adiabatically. However, in this case there is no gap and the issue is more delicate. The argument would be strengthened if the picture of the $\nu = 1/2$ state given above could be shown to generalize to the $\nu = 1/3$ state on the thin torus, in which case it should be relevant also for the bulk $\nu = 1/3$ state. The mapping of the low-energy sector at $\nu = 1/2$ onto an $s = 1/2$ XY-spin chain would then presumably generalize into a mapping of $\nu = 1/3$ onto an $s = 1$ chain. In passing, we note that this suggests that the Haldane conjecture for the gaps in spin chains [21] might apply to the two-dimensional electron gas in a strong magnetic field.

The thin torus, or cylinder for that matter, with a magnetic field perpendicular to its surface is probably not experimentally accessible. Thus, the experimental consequences of the results in this Letter presumably depend on whether the results are applicable, *mutatis mutandis*, to the bulk case as we conjecture. Our conjecture implies that the $\nu = 1/2$ state is a one-dimensional Luttinger liquid rather than a two-dimensional Fermi theory. We predict that this leads to observable effects in the bulk $\nu = 1/2$ system, such as nonlinear I-V characteristics determined by the Luttinger liquid interaction parameter.

If our conjecture is correct, then it should be possible to understand the experimental results that are successfully explained by the standard composite fermion theory [4], such as the ballistic transport and the surface acoustic wave results. We note that the appearance of low-energy excitations that are neutral, and hence do not couple to the magnetic field, is consistent with the ballistic transport results.

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