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Universality Class of the Fiber Bundle Model on Complex Networks

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We investigate the failure characteristics of complex networks within the framework of the fiber bundle model subject to the local load sharing rule in which the load of the broken fiber is transferred only to its neighbor fibers. Although the load sharing is strictly local, it is found that the critical behavior belongs to the universality class of global load sharing where the load is transferred equally to all fibers in the system. From the numerical simulations and the analytical approach applied to the microscopic behavior, it is revealed that the emergence of a single dominant hub cluster of broken fibers causes the global load sharing effect in the failure process.

The fiber bundle model (FBM) has been studied for many years in order to explain a variety of failure phenomena caused by cascades [1]. In the FBM, composed of N heterogeneous fibers put on a lattice, a fiber at the vth site is broken if the load σ_v is larger than the threshold value σ_v^{th} assigned to the fiber following a given probability distribution function. When the fiber is broken, the load which was supported by the broken fiber is shared among intact fibers following a load sharing rule. The two most frequently studied rules are global load sharing (GLS) [2], in which the load of a broken fiber is equally shared with all intact fibers in the whole system, and local load sharing (LLS) [3], which allows only the nearest intact fibers to carry the load of a broken fiber.

Depending on which load sharing rule is used, the FBM has been shown to exhibit totally different behaviors: For GLS, the FBM has been found to have a phase transition toward the global failure as the external load per fiber ($\bar{\sigma}$) is increased beyond the nonzero critical point $\bar{\sigma}_c$ [4]. It has been known that the avalanche size distribution for GLS follows the power-law form with the universal exponent $-5/2$ [5,6], and the universality class has been identified from the measurement of critical exponents [7]. In contrast, the FBM under the LLS rule has been shown to belong to a completely different universality class; i.e., the critical value of the load approaches zero as N is increased following the form $\bar{\sigma}_c \sim 1/\ln(N)$ [8,9]. Recently, the transition between the GLS and the LLS regime has been studied using modified load sharing rules [10].

Until very recently, most studies of the FBM have been performed on regular lattice structures. In a general perspective beyond the fracture of material, however, cascading failure triggered by overloading happens also in real-world network systems [11,12]. For instance, the recent blackout in the United States and Canada was caused by the cascading breakdown of elements through a power grid. The scenario of a major blackout is very similar to the idea of the FBM; once one element fails, then the

neighbor of the element fails by the increased load from the failure.

In this Letter, we numerically and analytically study the FBM subject to the LLS rule on various network structures: i.e., the Erdős-Rényi (ER) model of a random network [13], the Watts-Strogatz (WS) model of a small-world network [14], and the static model of a scale-free network [15]. Even though the model studied in this work strictly obeys the LLS rule, it is found that the FBM on complex networks exhibits completely different universality: the critical behavior, the avalanche size distribution, and the form of failure probability function coincide with those of the FBM under the GLS rule.

First, we briefly describe the FBM under the LLS rule on complex networks. A version of blackout scenarios for cascading breakdown of power plants from overloading is very intuitive to understand how the FBM on complex networks works. Fibers attached to the vertices of underlying networks act as power plants, and the external load on fibers can be regarded as the demand for electric power. If the demand for electric power exceeds the capacity of a power plant, the power plant gets disconnected from the network and the demand is transferred to neighboring power plants through the transmission lines, the edges of the network. In this model, the underlying network is rigid while the fibers attached to vertices are damaged.

The local load transfer of broken fibers through the edges of the underlying network is governed by the LLS rule. Under a nonzero external load $N\bar{\sigma}$, the actual load σ_v of the intact fiber v is given by the sum of $\bar{\sigma}$ and the transferred load from neighboring broken fibers. To systematically handle the local load transfer from broken fibers to intact fibers, we define the load concentration factor $K_v \equiv \sigma_v/\bar{\sigma}$ as $K_v = 1 + \sum'_j m_j/k_j$, where the primed summation is over the cluster of broken fibers directly connected to v, m_j is the number of broken fibers in the cluster j, and k_j is the number of intact fibers directly connected to j. A simple example is shown in Fig. 1, which
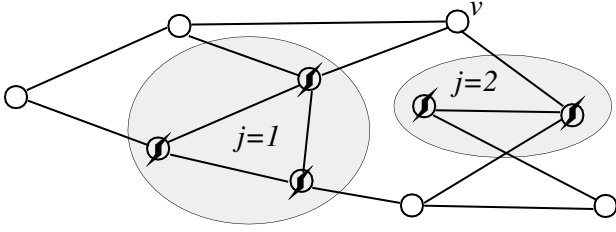
FIG. 1. The LLS rule applied for the FBM on a complex network. The vertices bound to broken fibers and intact fibers are denoted by the broken symbols and the open circles, respectively.
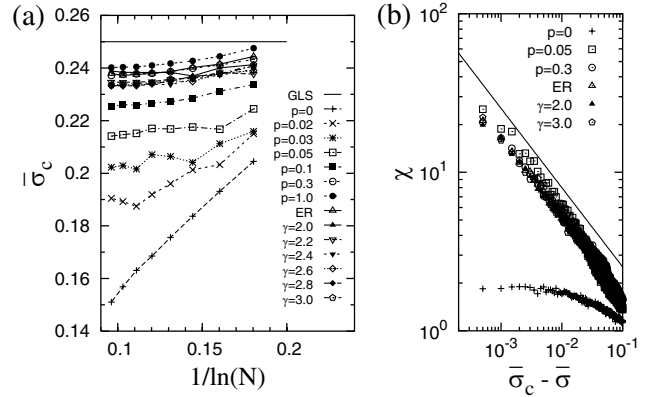

FIG. 2. (a) The system size (N) dependence of critical points ($\bar{\sigma}_c$) for various networks with $N = 2^8, 2^9, \ldots, 2^{15}$ vertices. (b) The susceptibility for the networks with $N = 2^{14}$. p and γ are the rewiring probability in the WS networks and the exponent of degree distribution $P(k) \sim k^{-\gamma}$ in the static model [15], respectively. The data points are obtained from the averages over 10^4 (10^3 for $N = 2^{15}$) ensembles.

contains two clusters of broken fibers, $m_1 = 3$, $k_1 = 4$ and $m_2 = 2$, $k_2 = 3$. If $\sigma_v^{\text{th}} < (1 + 3/4 + 2/3)\bar{\sigma}$, the fiber at v will be broken and join the clusters of broken fibers. We note that this is the generalization of the 1D model ($k_j = 2$) [16,17].

To explore all values of the external load $N\bar{\sigma}$, we increase $\bar{\sigma}$ quasistatically starting from zero. In a finite-sized system, the infinitesimal increment δ of $\bar{\sigma}$ is expressed as $\delta = \min_v[\frac{\sigma_v^{\text{th}}}{K_v} - \bar{\sigma}]$ with the minimization ($\min_v$) for all intact fibers. This condition is equivalent to the increase of $\bar{\sigma}$ just enough to break only the weakest intact fiber, which is the minimal condition to trigger an avalanche.

Our numerical scheme is as follows: The threshold value of the load $\sigma_v^{\text{th}} \in [0, 1]$ is assigned to each fiber following the uniform distribution function [18]. Start from $\bar{\sigma} = 0$ and repeat the following two steps: (i) Increase $\bar{\sigma}$ by the infinitesimal increment δ. (ii) Following the LLS rule, break the fibers with $\sigma_v^{\text{th}} < K_v\bar{\sigma}$ iteratively until no more fibers break. For each increment of $\bar{\sigma}$, the size $s(\bar{\sigma})$ of the avalanche is defined as the number of broken fibers triggered by the increment. The surviving fraction $x(\bar{\sigma})$ of fibers is the ratio of the number of remaining intact fibers to N when the external load reaches $N\bar{\sigma}$, and thus is written as

$$x(\bar{\sigma}) = 1 - \frac{1}{N}\sum_{\sigma < \bar{\sigma}} s(\sigma). \qquad (1)$$

We also measure the response function χ, or the generalized susceptibility, by using

$$\chi(\bar{\sigma}) = \left|\frac{dx}{d\bar{\sigma}}\right| = \frac{1}{N\Delta}\sum_{\bar{\sigma} < \sigma < \bar{\sigma}+\Delta} s(\sigma), \qquad (2)$$

where we choose the small enough value $\Delta = 0.0005$ for the numerical differentiation. The critical value $\bar{\sigma}_c$ of the external load, which is one of the key quantities of interest, is defined from the condition of the global breakdown $x(\bar{\sigma}_c) = 0$.

Figure 2 displays the critical value $\bar{\sigma}_c$ and the susceptibility χ for the FBM under the LLS rule on various network structures, such as the local regular network, the WS network, the ER network, and the scale-free networks. Strikingly, we find that the critical behavior of the FBM on complex networks is completely different from that on a regular lattice. More specifically, while $\bar{\sigma}_c$ for the FBM on a local regular network vanishes in the thermodynamic limit and is described by $\bar{\sigma}_c \sim 1/\ln(N)$ for finite-sized systems (see the curve for $p = 0$ in Fig. 2, corresponding to the WS network with the rewiring probability $p = 0$), $\bar{\sigma}_c$ for *all* networks except for the local regular one does not diminish but converges to a nonzero value as N is increased. Moreover, the susceptibility diverges at the critical point following $\chi \sim (\bar{\sigma}_c - \bar{\sigma})^{-0.5}$, regardless of the networks, which is again in sharp contrast to the local regular network [see Fig. 2(b)]. The critical exponent 0.5 clearly indicates that the FBM under the LLS rule on complex networks belongs to the same universality class as that of the GLS regime [7] although the load sharing rule is strictly local.

The evidences that the LLS model on complex networks belongs to the universality class of the GLS model is also found in the avalanche size distribution $P(s)$: Unanimously observed power-law behavior $P(s) \sim s^{-5/2}$ in Fig. 3(a) for all networks except for the local regular one (the WS network with $p = 0$) is in perfect agreement with the same behavior for the GLS case [5]. On the other hand, the LLS model for a regular lattice has been shown to exhibit completely different avalanche size distribution [6,9].

The failure probability $F(\bar{\sigma})$ is defined as the probability that the whole system is broken when an external load $\bar{\sigma}$, or less, is applied. In the LLS regime, the failure probability was studied to test the weak-link hypothesis [17,19,20]. In the test for the Weibull form, one can see in Fig. 3(b) that $F(\bar{\sigma})$'s for complex networks fall on a common line which coincides with $F(\bar{\sigma})$ for the well-known GLS case, which
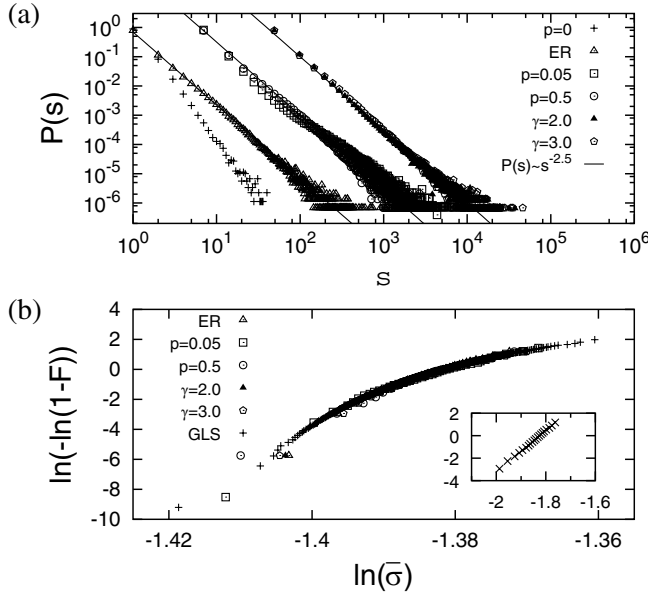
FIG. 3. (a) The avalanche size distribution $P(s)$ and (b) the test of failure probability F for the Weibull form. The solid lines in (a) indicate the result for the GLS regime having exponent $-5/2$. The inset of (b) shows the result for the LLS regime on the regular network ($p = 0$), which has a clearly different form from the others.

FIG. 4. (a) The largest cluster size m_L in the failure process versus the external load $\bar{\sigma}$. (b) The average distance $\bar{d}$ from the DLC versus the surviving fraction x. (c) The fraction $n(x)$ of nearest intact fibers of the DLC as a function of the surviving fraction x. The symbols and the lines represent the numerical data and their fits to the curve of Eq. (5), respectively. The data points are obtained for the networks with $N = 2^{14}$ and averaged over 10^4 ensembles.

is very much different for LLS on regular lattices [17,19,20] [see the inset of Fig. 3(b)]. Consequently, we again confirm that the LLS model on complex networks belongs to the same universality class as that of GLS.

What makes the LLS model on a complex network have identical critical behaviors of the GLS model? In order to answer this question, it is helpful to investigate the microscopic details of the failure process. As the external load increases, the clusters of broken vertices form, grow, and merge into larger clusters, ultimately resulting in the emergence of a dominantly large cluster (we call it DLC henceforth). In case of a regular lattice, all small clusters of broken fibers are roughly in an equal condition because of the underlying regular (and thus spatially uniform) topology. As the external load increases, all small clusters grow at roughly the same rate, and thus the DLC emerges abruptly. In contrast, the emergence of a DLC on complex networks is a gradual process because the DLC is formed at an early stage of loading and keeps growing continuously, as shown clearly in Fig. 4(a). The above observation indicates that the growth of the DLC plays a dominant role in the failure of fibers on complex networks.

Together with the early emergence and the gradual growth of the DLC, the small-world behavior [14] in complex networks provides a reasonable qualitative explanation of the GLS-like behavior of the LLS model on complex networks. More precisely, the small-world effect causes the average distance from the DLC to intact fibers to decay fast towards the value below two as the size of the
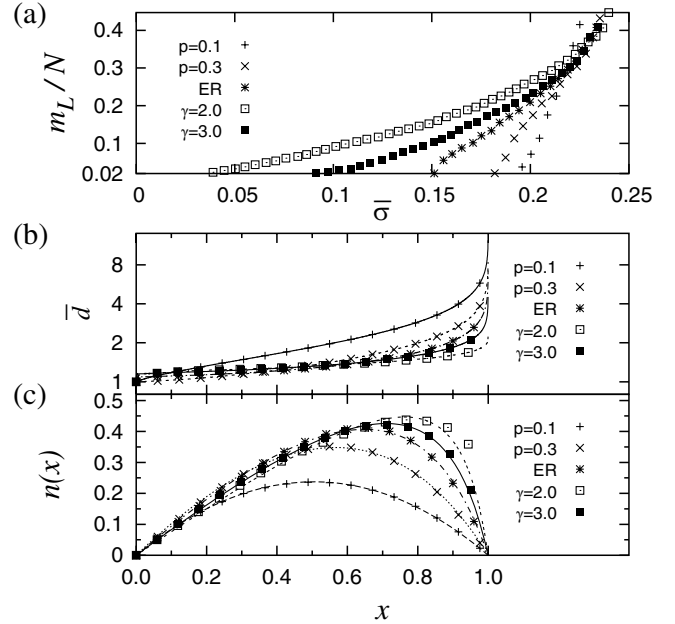
DLC increases (or the surviving fraction x decreases) [see Fig. 4(b)], which implies that most intact fibers are very closely located to the DLC and thus the system behaves similarly to the GLS model with all fibers separated by the unit distance from each other.

Finally, we apply the mean-field theory to the growth of the DLC. Let us assume the situation that the system is at the tth step of the load transfer from the DLC. The surviving fraction is written as $x^{(t)}$, and thus the cluster is composed of $N(1 - x^{(t)})$ broken vertices (we assume that there exists only one cluster) and has $k^{(t)}$ nearest neighbor intact vertices. Following the LLS rule, the load $\sigma^{(t)}$ of the nearest intact vertices of the cluster satisfies the load conservation condition, which yields

$$\sigma^{(t)} = \frac{1 + n^{(t)} - x^{(t)}}{n^{(t)}} \bar{\sigma}, \qquad (3)$$

where $n^{(t)} \equiv k^{(t)}/N$. Assuming that the load threshold σ^{th} of a fiber is randomly distributed to the whole system following the cumulative distribution $P_{\mathrm{cum}}(\sigma^{\mathrm{th}})$, we establish the recursion equation for x,

$$x^{(t+1)} = x^{(t)} - n^{(t)} \left[1 - \frac{1 - P_{\mathrm{cum}}(\sigma^{(t)})}{x^{(t)}} \right]. \qquad (4)$$

In order to solve the equations, we have to know the functional form of $n^{(t)}$, which is difficult to determine analyti-

cally. Instead, we assume the simple rational functional form with two fitting parameters $a, b \in [0, 1]$.

$$n^{(t)} \simeq bx^{(t)}(1 - x^{(t)})/(1 - ax^{(t)}), \qquad (5)$$

which is motivated as a generalization of the GLS form (corresponding to $a = b = 1$). The numerical data of $n^{(t)}$ fit very well to the form (5) as shown in Fig. 4(c).

When the failure stops propagating, the fixed point of the dynamics described by Eq. (4) satisfies $x^{(t+1)} = x^{(t)} \equiv x^*$. The uniform threshold distribution $P_{\text{cum}}(\sigma) = \sigma$, together with Eqs. (3)–(5), results in

$$\bar{\sigma}(x^*) = \frac{bx^*(1 - x^*)}{1 + (b - a)x^*}. \qquad (6)$$

The critical value $\bar{\sigma}_c$ of the external load and the surviving fraction at $\bar{\sigma}_c$ are then easily obtained from the maximum of $\bar{\sigma}(x^*)$, yielding

$$\bar{\sigma}_c = bx_c^{*2}, \qquad x_c^* = \frac{-1 + \sqrt{1 + b - a}}{b - a}. \qquad (7)$$

Near the GLS regime, $b = 1$ and $a = 1 - \epsilon$, one obtains

$$\bar{\sigma}_c \simeq 1/4 - \epsilon/8, \qquad x_c^* \simeq 1/2 - \epsilon/8, \qquad (8)$$

which indicates that the solution for the GLS regime is revisited at $\epsilon = 0$, in a good agreement with the numerical simulation where $\bar{\sigma}_c$ has always been found to be smaller than the GLS value $\bar{\sigma}_c = 1/4$ (see Fig. 2). In addition, combining Eqs. (3) and (6), we obtain the critical behavior of x^* near $\bar{\sigma}_c$,

$$\chi = \left| \frac{dx^*}{d\bar{\sigma}} \right| \simeq (\bar{\sigma}_c - \bar{\sigma})^{-1/2}, \qquad (9)$$

confirming the universal exponent -0.5 for the GLS regime.

In conclusion, we have investigated the FBM on various complex networks, including the WS network, the ER random network, and the scale-free network. From numerical simulations and an analytical approach, it has been found that, although the LLS rule is strictly applied, the critical behavior of failure exhibits the characteristics of GLS: The critical value of the external load is finite in the thermodynamic limit; the divergence of the susceptibility is described by the same critical exponent as in GLS; the avalanche size distribution and the statistics of failure display the unique behavior of the GLS regime.

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