Power of an Axisymmetric Pulsar

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Certain exact properties of the stationary force-free magnetosphere of an axisymmetric pulsar are obtained. In particular, it is shown that a magnetic separatrix has an inclination angle of 77.3° to the equatorial plane. The electromagnetic field has an $R^{-1/2}$ singularity inside the separatrix near the light cylinder. A numerical simulation of the magnetosphere which crudely reproduces these properties is presented. The numerical results are used to estimate the power of an axisymmetric pulsar: $L = (1 \pm 0.1)\mu^2\Omega^4/c^3$. Thus, the background magnetic configuration, on which all the puzzling pulsar phenomena are taking place, is now known with some confidence.

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I. Introduction.—A magnetic dipole μ rotating at angular velocity Ω loses energy at a rate $L = (2/3)\mu^2\sin^2\theta\Omega^4/c^3$, where θ is the angle between the rotation axis and the dipole. This energy is emitted in a form of electromagnetic radiation with frequency Ω.

It is thought that a magnetized neutron star rotating around its magnetic axis loses energy at about the magnetodipole rate $L \sim \mu^2\Omega^4/c^3$, even though $\theta = 0$ in this case, and classical magnetodipole formula predicts zero radiated power. It is also known that such neutron stars, known as pulsars, actually emit radiation at frequencies much greater than Ω—all the way from radio to gamma rays in some cases. The energy loss from an aligned rotator is possible because the star creates free charges that form a magnetosphere with nonzero Poynting flux along open field lines—the well-known prediction of Goldreich and Julian [1]. We confirm this prediction in this Letter and *calculate* the power of axisymmetric pulsars $L = (1 \pm 0.1)\mu^2\Omega^4/c^3$.

The pulsar power formula is of great importance in astrophysics. In particular, one estimates the magnetic fields of neutron stars by measuring the pulsar spin-down rate and then applying the magnetodipole formula. But [1] does not solve the magnetosphere equation. Not only the dimensionless coefficient in the power was unknown, it was not even proven that the pulsar power is independent of the radius of the neutron star.

Another reason for a rigorous analysis of the axisymmetric pulsar is the puzzle of pulsar luminosity. Pulsar radio emission is characterized by very high brightness temperatures, and is thought to be coherent. The mechanism of coherent radio emission is unknown. Clearly, the first step in understanding radio emission is the calculation of the unperturbed background—the stationary force-free magnetosphere.

The first calculation of the shape of the force-free axisymmetric pulsar magnetosphere was performed by Contopoulos, Kazanas, and Fendt [2]. This important paper demonstrated that a stationary solution does exist, and the power of the pulsar must be close to the magnetodipole value. But, as we will show, the resolution of [2] was inadequate. Because of poor resolution, [2] missed a potentially important property of the pulsar magnetosphere—the $R^{-1/2}$ singularity of electromagnetic field in the vicinity of the critical circle (the intersection of the light cylinder and the equatorial plane; see below).

In Sec. II, by solving the stationary force-free equations in the vicinity of the critical circle, we show that (i) the separatrix inclination angle is equal to 77.3° and (ii) the electromagnetic field has an $R^{-1/2}$ singularity near the critical circle inside the separatrix. Neither of these properties are seen in the numerical results of [2].

We therefore repeated the simulation of [2] and found the following (Sec. III). At numerical resolution similar to that of [2], our code reproduces most of their results. But at higher resolution, the separatrix steepens and fattens, and the singularity of the electromagnetic field inside the separatrix starts to develop. The numerical simulation crudely reproduces the predicted properties of the magnetosphere. We used our numerical results to estimate the power of an axisymmetric pulsar: $L = (1 \pm 0.1)\mu^2\Omega^4/c^3$.

Pulsar magnetosphere equation: In the appendix we describe force-free electrodynamics (FFE)—a remarkable version of plasma physics without any plasma properties appearing explicitly. There we also discuss the applicability of the force-free approximation. To finish the introduction we must give a brief derivation of the pulsar magnetosphere equation [3].

It is assumed that electromagnetic forces are much stronger than inertia, thus $\mathbf{j} \times \mathbf{B} + c\rho\mathbf{E} = 0$, or

$$\nabla \times \mathbf{B} \times \mathbf{B} + \nabla^2\phi\nabla\phi = 0. \quad (1)$$

Here E, B are electric and magnetic fields, ρ, j are charge and current density, and ϕ is the electrostatic potential; the fields are stationary. For axisymmetric fields, we represent the magnetic field by the toroidal component of the vector potential ψ/r and by the quantity $A \equiv 2I/c$, where I is the poloidal current:

$$\mathbf{B} = \left(-\frac{\psi_z}{r}, \frac{A}{r}, \frac{\psi_r}{r}\right), \quad (2)$$

in cylindrical coordinates r, z; subscripts mean partial derivatives. Using (2) in (1), one gets the pulsar magnetosphere equation [3]:

$$(1 - r^2)\left(\psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz}\right) - \frac{2}{r}\psi_r + F(\psi) = 0. \quad (3)$$

Here $F(\psi) \equiv A(\psi)\frac{dA(\psi)}{d\psi}$, $A(\psi)$ is an arbitrary function, $\phi = \psi$ follows from the boundary condition on the surface of the star, and we use dimensionless units

$$c = \Omega = \mu = 1. \quad (4)$$

The basic equation (3) must be solved with the small distance boundary condition

$$\psi \to \frac{r^2}{(r^2 + z^2)^{3/2}}, \qquad r, z \to 0, \quad (5)$$

which corresponds to the dipole field.

Michel [4] has found an exact solution of the magnetosphere equation for a magnetic monopole rather than a dipole star, proving that solutions of (3) which are smooth across the light cylinder ($r = 1$) do exist in some cases.

II. Near the singular circle.—The basic equation (3) can be solved in the vicinity of the singular circle, $|z| \ll 1$ and $|r - 1| \ll 1$. In this region, (3) can be approximated as

$$x(\psi_{xx} + \psi_{zz}) + \psi_x = \frac{1}{2}F, \quad (6)$$

where $x \equiv r - 1$.

We assume that there is a nonzero return current flowing along the separatrix (Fig. 1). This assumption will be confirmed by numerical simulations of Sec. III. Also, one can show that a zero return current assumption leads to the breakdown of the force-free approximation in the vicinity of the singular circle [5]. On the other hand, nonzero return current automatically gives $B > E$ in the vicinity of the critical circle, because near the light cylinder the magnitude of poloidal magnetic field is close to the magnitude of electric field. We have also checked that the numerical solution of Sec. III gives $B > E$ everywhere, not just near the critical point.

Let ψ_0 be the value of the potential on the separatrix, and $A_0 \equiv A(\psi_0 - 0)$. Then the return current is equal to $A_0/2$. From (6), we get the jump condition across the separatrix

$$(\nabla\psi)^2|_{\psi=\psi_0+0} - (\nabla\psi)^2|_{\psi=\psi_0-0} = -\frac{1}{2x}A_0^2. \quad (7)$$

In the closed line region, for $\psi > \psi_0$, we must therefore have $\psi \propto (-x)^{1/2}$. Thus electric and magnetic fields diverge as inverse square root in the vicinity of the singular circle. This is an admissible singularity, since the total energy of the fields remains finite.

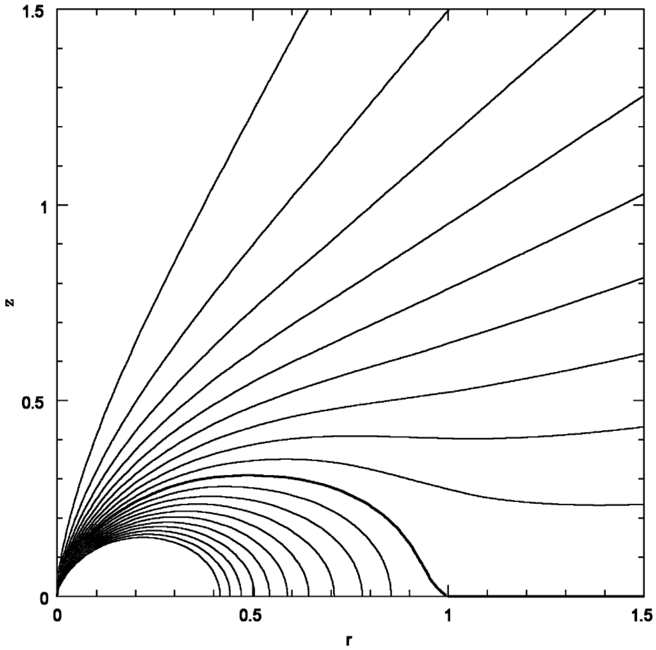

FIG. 1. Stationary force-free axisymmetric pulsar magnetosphere. The thick line shows the separatrix $\psi_0 = 1.27$. Thin lines correspond to ψ intervals of $0.1\psi_0$.

We can now find the leading order solution in the closed line region. We set

$$\psi = \psi_0 + R^{1/2}f(\theta), \quad (8)$$

where $x \equiv R\sin\theta$ and $z \equiv R\cos\theta$. Then (6) gives inside the separatrix

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{3}{4}f = 0. \quad (9)$$

Solving this ordinary equation numerically for $df/d\theta = 0$ at $\theta = -\pi/2$, we find $f(\theta = -0.222) = 0$. Thus the inclination angle of the separatrix is 77.3°.

Knowing the separatrix inclination angle, we can solve (6) in the open line region, too. We assume that in the open line region $\psi - \psi_0 = R^\alpha f(\theta)$, and correspondingly $F \propto (\psi_0 - \psi)^{1-(1/\alpha)}$. Then Eq. (6) reads

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \alpha(\alpha + 1)f = \frac{C}{\sin\theta}f^{1-(1/\alpha)}. \quad (10)$$

Here the two free parameters α and C should be adjusted so as to have (i) $f(-0.222) = 0$, (ii) $f(\pi/2) = 0$, and (iii) no singularity at $\theta = 0$. A numerical solution gives

$$\psi_0 - \psi \propto R^{2.4}, \qquad F(\psi) \propto -(\psi_0 - \psi)^{0.58}. \quad (11)$$

All these properties are roughly reproduced by the numerical simulation presented in Sec. III.

III. Axisymmetric pulsar magnetosphere.—Numerical solution of the axisymmetric pulsar equation (3) can be obtained in the following way [2,6]. One takes an arbitrary $F(\psi)$ and solves (3) in the inner ($r < 1$) and in the outer

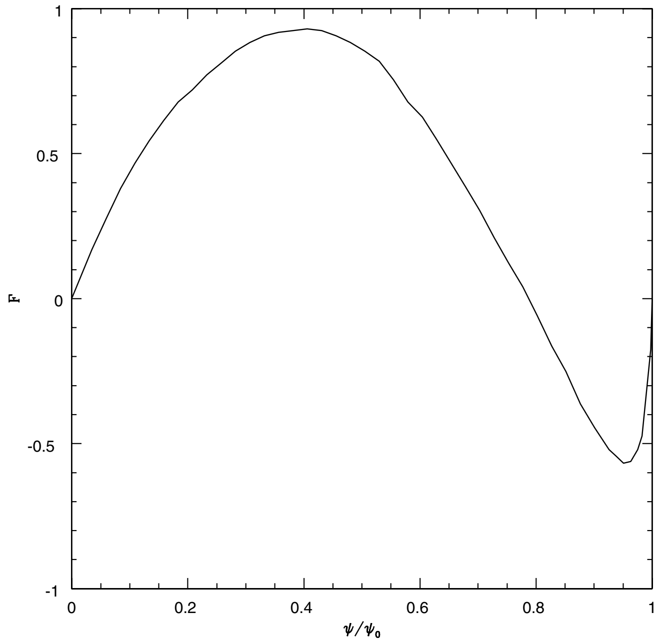

FIG. 2. The function F which makes the solution of the magnetosphere equation (3) smooth across the light cylinder.

$(r > 1)$ regions using appropriate boundary conditions [the boundary condition at the light cylinder being $\psi_r = F(\psi)/2$]. This, of course, does not give the true solution, because one gets $\psi(1 - 0, z) \neq \psi(1 + 0, z)$. One then makes a number of adjustments of $F(\psi)$, aimed at reducing the jump $\psi(1 - 0, z) - \psi(1 + 0, z)$ for all z. An important finding of [2] is that this procedure actually gives an everywhere smooth solution.

In our simulation we followed the same method. We were adjusting $F(\psi)$ until an acceptable solution was obtained. A solution was called acceptable if the following integral $\int_0^\infty dz[\psi(1 - 0, z) - \psi(1 + 0, z)]^2/(1 + z^2)$ was reduced to less then 10^{-7} (starting from ≈ 1 at $F \equiv 0$). Equation (3) was solved by a simple relaxation method. Simultaneously with the relaxation, the adjustment of F was carried out, in a way similar to that of [2]. After an acceptable solution was obtained, the adjustment of F was stopped, while the relaxation was carried out for a sufficient number of steps to ensure that our $\psi(r, z)$ does solve (3) for the obtained $F(\psi)$.

The full function F_{full} which must be used in (3) consists of the regular piece F (the one shown in Fig. 2) and the delta-function piece $F_\delta(\psi) = -(\int F)\delta(\psi - \psi_0)$. In the numerical simulation the delta-function piece was smoothed over the ψ interval $[\psi_0, \psi_0(1 + d)]$. We modeled the delta function by a parabola, but we have checked that a constant gives the same result. Most importantly, the simulation was carried out for different values of d. For a given spatial resolution, the numerical scheme works well only for large enough d. Small values of d require good resolution. The figures show the case $d = 0.03$. Table I lists some of the simulations that were carried out. The independence of numerical results on the radius of the star, for small radii of the star, and on the outer boundaries location, for distant boundaries, was checked.

The power of the pulsar, which is proportional to the spin-down rate, is obtained by integrating the Poynting flux over an arbitrary sphere. One gets the dimensionless power

$$L = \int_0^{\psi_0} d\psi A(\psi), \tag{12}$$

where $A(\psi) = [2 \int_0^\psi d\psi' F(\psi')]^{1/2}$ is obtained from the regular part of F. From Table I we estimate the power of an axisymmetric pulsar $L = (1 \pm 0.1)\mu^2\Omega^4/c^3$.

Our numerical solution reproduces the features predicted in Sec. II in the following sense: (i) the inclination angle increases with decreasing d and the extrapolated value of inclination is about 70°, (ii) the maximum of the magnetic field in the inner part of the separatrix becomes more pronounced with decreasing d, and (iii) the function F demonstrates a singularity similar to (11).

However, we have not accurately reproduced either of the four numbers given Sec. II. A really good numerical solution should show (i) the 77° separatrix inclination, (ii) the $\psi - \psi_0 \propto R^{1/2}$ singularity in the inner region, (iii) the $F(\psi) \propto -(\psi_0 - \psi)^{0.58}$ singularity, and (iv) the $\psi_0 - \psi \propto R^{2.4}$ singularity in the outer region. Until such a solution is obtained, one cannot be really sure that a stationary force-free pulsar magnetosphere exists, and one

TABLE I. Simulation results.

Resolution	δ function width d	Separatrix value ψ_0	Separatrix inclination (°)	Power
200×100	0.02	1.27	63	1.01
200×100	0.03	1.27	60	1.02
200×100	0.04	1.30	56	1.06
200×100	0.06	1.32	50	1.10
200×100	0.08	1.35	48	1.15
200×100	0.1	1.38	48	1.19
100×50	0.1	1.40	42	1.27
100×50	0.06	1.34	48	1.17

cannot be really sure that the power of an axisymmetric pulsar is $L = (1 \pm 0.1)\mu^2\Omega^4/c^3$.

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Appendix: Force-free electrodynamics.—(This appendix contains a large excerpt from Ref. [7].) Force-free electrodynamics is applicable if electromagnetic fields are strong enough to produce pairs and baryon contamination is prevented by strong gravitational fields [8]. Pulsars, Kerr black holes in external magnetic fields, relativistic accretion disks, and gamma-ray bursts are the astrophysical objects whose luminosity might come originally in a pure electromagnetic form describable by FFE.

FFE is classical electrodynamics supplemented by the force-free condition:

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \tag{A1}$$

$$\partial_t \mathbf{E} = \nabla \times \mathbf{B} - \mathbf{j}, \tag{A2}$$

$$\rho \mathbf{E} + \mathbf{j} \times \mathbf{B} = 0. \tag{A3}$$

$\nabla \cdot \mathbf{B} = 0$ is the initial condition. The speed of light is $c = 1$; $\rho = \nabla \cdot \mathbf{E}$ and $\mathbf{j}$ are the charge and current densities multiplied by 4π. The electric field is everywhere perpendicular to the magnetic field, $\mathbf{E} \cdot \mathbf{B} = 0$. The electric field component parallel to the magnetic field should vanish because charges are freely available in FFE. It is also assumed that the electric field is everywhere weaker than the magnetic field, $E^2 < B^2$. Then Eq. (A3) means that it is always possible to find a local reference frame where the field is a pure magnetic field, and the current is flowing along this field. FFE is Lorentz invariant.

Equation (A3) can be written in the form of Ohm's law. The current perpendicular to the local magnetic field can be calculated from Eq. (A3). The parallel current is determined from the condition that electric and magnetic fields remain perpendicular during the evolution described by the Maxwell equations (A1) and (A2). We thus obtain the following nonlinear Ohm's law:

$$\mathbf{j} = \frac{(\mathbf{B} \cdot \nabla \times \mathbf{B} - \mathbf{E} \cdot \nabla \times \mathbf{E})\mathbf{B} + (\nabla \cdot \mathbf{E})\mathbf{E} \times \mathbf{B}}{B^2}. \tag{A4}$$

Equations (A1), (A2), and (A4) form an evolutionary system (initial condition $\mathbf{E} \cdot \mathbf{B} = 0$ is assumed). It therefore makes sense to study stability of equilibrium electromagnetic fields in FFE. One can also study linear waves and their nonlinear interactions in the framework of FFE [9].

One can introduce a formulation of FFE similar to magnetohydrodynamics (MHD); then we can use the familiar techniques of MHD to test stability of magnetic configurations. To this end, define a field $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$, which is similar to velocity in MHD. Then $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and Eq. (A1) becomes the "frozen-in" law

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{A5}$$

From $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$, and from Eqs. (A1)–(A3), one obtains the momentum equation

$$\partial_t (B^2 \mathbf{v}) = \nabla \times \mathbf{B} \times \mathbf{B} + \nabla \times \mathbf{E} \times \mathbf{E} + (\nabla \cdot \mathbf{E})\mathbf{E}, \tag{A6}$$

where $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$. Equations (A5) and (A6) are the usual MHD equations except that the density is equal to B^2 and there are order v^2 corrections in the momentum equation.

We must mention that the applicability of FFE to pulsars can be questioned [10]. If charges are not freely available, some regions of the pulsar magnetosphere might exist that should be described by a vacuum rather than force-free electrodynamics. While we cannot offer a real description of the creation of the space charge, a simple energy estimate shows that the system might find a way to put the entire magnetosphere into the force-free regime. Indeed, the pulsar luminosity is $L \sim B^2 R^6 \Omega^4/c^3$, where B is the magnetic field and R is the radius. The number density of charged particles is $n \sim \Omega B/(ce)$. The associated energy density is nmc^2, and the associated power in particles is $L_p \sim nmc^3 R^2$. The ratio $L_p/L \sim mc^5/(e\Omega^3 B R^4)$ is a very small number everywhere in the magnetosphere. Thus, energywise, charges are indeed freely available. With only a tiny fraction of the pulsar luminosity channeled into the charge production, the star will be able to put the entire magnetosphere into the force-free state. The real mechanisms for populating the magnetosphere are of course of great importance, but these probably include complex interactions of the pulsar radiation, high-energy electrons and positrons, the surface of the neutron star, the large-scale and turbulent electromagnetic fields—should be difficult to decipher. But FFE might well turn out to be a good approximation to reality.

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