Microscopic Verification of Topological Electron-Vortex Binding in the Lowest Landau-Level Crystal State

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When two-dimensional electrons are subjected to a very strong magnetic field, they are believed to form a triangular crystal. By a direct comparison with the *exact* wave function, we demonstrate that this crystal is not a simple Hartree-Fock crystal of electrons but an inherently quantum mechanical crystal characterized by a nonperturbative binding of quantized vortices to electrons. It is suggested that this has qualitative consequences for experiment.

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The quantum mechanical behavior of solids has been the subject of much investigation in the context of the quantum solids of ^{3}He and ^{4}He [1]. There has been a revival of interest in this topic due to the recently reported "supersolid" phase of ^{4}He [2], resulting from a significant overlap between the wave functions of neighboring atoms [3]. This Letter demonstrates that the lowest Landau-level (LL) crystal of electrons provides another example of an inherently quantum mechanical crystal.

Much work has been done on the lowest LL crystal for over two decades. When a two-dimensional electron system is exposed to a magnetic field (B), the kinetic energy is quantized into LLs. The number of occupied LLs is called the filling factor, $\nu = \rho hc/eB$, ρ being the two-dimensional electron density. At sufficiently strong magnetic fields, when all electrons fall into the lowest LL ($\nu < 1$), the kinetic energy is no longer relevant, and the nature of the state is determined solely by the Coulomb interaction. Following Wigner [4], the dominance of the interaction energy can be expected to produce an electron crystal. For a range of filling factors the system condenses into a quantum liquid, characterized by dissipationless transport and precisely quantized plateaus of Hall resistance [5]. There are strong indications, however, that a crystal occurs at sufficiently low ν [6], and its properties have been probed experimentally by transport [7–11] and by electromagnetic waves [12–17], as well as theoretically [18–30].

Certain microscopic wave functions [31,32] are known to provide a good account of the fractional-quantum-Hall-effect (FQHE) liquid, as ascertained from comparisons with exact solutions known for systems containing a finite number of electrons, but become progressively worse with decreasing ν. A Hartree-Fock wave function describing an "electron crystal" (EC) provides a better approximation for the ground state at low ν. However, the Hartree-Fock crystal is not particularly good either, which has raised questions regarding the true nature of the crystal. A very interesting proposal suggests that the physics underlying the FQHE liquid is also operative in the crystal phase [27,28]. Yi and Fertig [27] have shown that a variational wave function in which vortices are bound to electrons has lower energy than the earlier Lam-Girvin [22] wave function in the filling factor range $0.1 < \nu < 0.2$. Narevich, Murthy, and Fertig [28] have used a Hamiltonian formulation of composite fermions to estimate gaps and shear modulus on either side of the $\nu = 1/5$ quantum Hall state.

The notion of binding of quantized vortices to electrons in the lowest LL crystal, if confirmed, would indicate the formation of a quantum crystal, given that vortices are inherently quantum mechanical objects. While the consequences of the quantum mechanical nature of such a crystal ought to be evaluated and tested by experiment, rigorous and unbiased theoretical tests of electron-vortex binding are possible because of the fortunate feature that the *exact* ground state wave function can be obtained, for finite systems, by a brute force numerical diagonalization for a wide range of ν in the crystal phase. The principal result of this work is to show that a wave function for the composite-fermion (CF) crystal, the composite fermion being the bound state of an electron and an even number of vortices, is extremely accurate at low ν—more accurate than the accepted FQHE wave functions for the liquid phase—thus establishing that the real crystal indeed has vortices bound to electrons. We also determine the parameter range where the CF crystal occurs. One might have expected the physics of the liquid to carry over into the crystal phase in the proximity of the phase boundary separating the liquid and the crystal, but our calculations indicate that the CF crystal is realized even deep inside the crystal phase, down to the lowest ν considered below.

The wave packet for an electron in the lowest LL localized at $\boldsymbol{R} = (X, Y)$ is given by [20]

$$\phi_{\boldsymbol{R}}(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{4}(\boldsymbol{r} - \boldsymbol{R})^2 + \frac{i}{2}(xY - yX)\right], \quad (1)$$

where the magnetic length, $l_0 = \sqrt{\hbar c/eB}$, has been taken as the unit of length. The wave function for the EC is constructed by placing electrons on a triangular lattice $\boldsymbol{R}_j$, the lowest energy solution for the classical problem, and then antisymmetrizing the product [20]:

$$\Psi^{\mathrm{EC}} = \frac{1}{\sqrt{N!}} \sum_P \epsilon_P \prod_{j=1}^{N} \phi_{\boldsymbol{R}_j}(\boldsymbol{r}_{Pj}), \tag{2}$$

where the sum is over all permutations P and ϵ_P is $+1$ for even permutations and -1 for odd permutations. With the lattice constant $a = (4\pi/\sqrt{3}\nu)^{1/2} l_0$, the overlap integral between nearest neighbor electron wave functions [20], $\exp(-a^2/2l_0^2) = \exp(-3.627/\nu)$, decays rapidly with decreasing ν. We work with the symmetric gauge, $\boldsymbol{A} = (B/2)(-y, x, 0)$, for which the total angular momentum L is a good quantum number. Because the wave function Ψ^{EC} is not an eigenstate of angular momentum, we follow the method of Yannouleas and Landman [33] to project it onto a definite L, denoting the resulting wave function Ψ_L^{EC}. Such projection amounts to creating a rotating crystal, implying that the crystalline structure is not apparent in the density but in the pair correlation function. The explicit expression for Ψ_L^{EC} is given in Ref. [33].

Following the standard procedure of the CF theory [32,34], we construct the following wave function:

$$\Psi_L^{2p\mathrm{CFC}} = \prod_{j<k} (z_j - z_k)^{2p} \Psi_{L^*}^{\mathrm{EC}}, \tag{3}$$

$$L^* = L - pN(N-1). \tag{4}$$

It is interpreted as a CF crystal (CFC), because the Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$ binds $2p$ quantized vortices to each electron in Ψ^{EC} to convert it into a composite fermion; the composite fermions of different flavors are denoted by ^{2p}CF, and their crystals by ^{2p}CFC. We next proceed to compare $\Psi^{^{2p}\mathrm{CFC}}$ with exact wave functions. The latter can be obtained (using the Lanczos method) for up to $N = 7$ particles in the low-ν region of interest. We present below detailed results for $N = 6$; the study of $N = 5$ and $N = 7$ particles is consistent with our conclusions below. The filling factor of the finite system is defined by the expression $\nu = N(N-1)/2L$, which gives the correct value of ν for a uniform density state in the thermodynamic limit. For $N = 6$, the lowest energy classical configuration has one particle at the center, with the remaining five forming a ring around it [35]. The wave functions $\Psi^{^{2p}\mathrm{CFC}}$ for $2p \neq 0$ have rather complicated correlations built into them, but the interaction energy per particle,

$$V = \frac{1}{N} \frac{\langle \Psi_L^{^{2p}\mathrm{CFC}} | \sum_{j<k} \frac{e^2}{\epsilon r_{jk}} | \Psi_L^{^{2p}\mathrm{CFC}} \rangle}{\langle \Psi_L^{^{2p}\mathrm{CFC}} | \Psi_L^{^{2p}\mathrm{CFC}} \rangle}, \tag{5}$$

can be evaluated by the Metropolis Monte Carlo method at least for many large values of $2p$ (the computation time increases rapidly as $2p$ is reduced). The total energy also has contributions from electron-background and background-background interactions, but these terms are the same for different crystal wave functions for a given L, so they are not relevant for comparisons.

Figure 1 shows the correlation energy of the optimal CF crystal, defined as the deviation of its energy from that of the uncorrelated electron crystal. (The computationally accessible range of $2p$ allows us to determine the minimum CFC energy. The only exceptions are the largest two values of L, where we show the energy at the smallest $2p$ studied; the minimum CF energy may be still lower here.) The exact correlation energy is also shown for $L \leq 145$. For $L > 145$, the dimension of the Fock space (D) is too large for an exact treatment. As an independent reference point, we obtain an accurate approximation to the exact energy by the method of "CF diagonalization," wherein the Coulomb Hamiltonian is diagonalized in a correlated CF basis, the dimension of which is much smaller than the dimension of the full basis needed for the exact state; gradually increasing the basis size gives an increasingly better approximation [36]. Figure 1 quotes $V_{\mathrm{CF}}^{(2)}$ (using the notation in Ref. [36]), obtained with a correlated CF basis of dimension 150. $V_{\mathrm{CF}}^{(2)}$ has been shown to be very precise [36]: for the six particle system it is within 0.02% of the exact energy for $L \leq 145$, and we expect a similar level of accuracy for higher L as well.

The minimum energy for all L is obtained at a nonzero value of $2p$, which establishes that the CF crystal provides

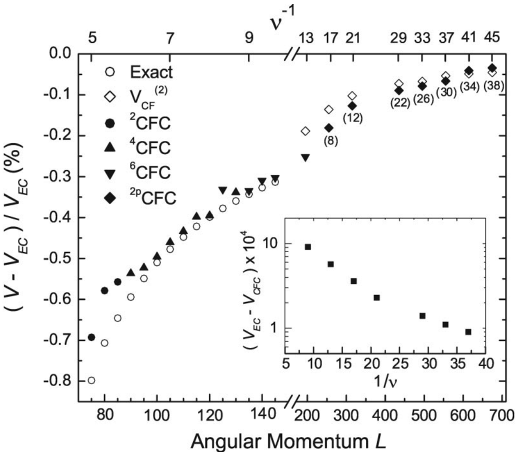

FIG. 1. The correlation energy of the optimal CF crystal, i.e., the percent of deviation of its Coulomb energy from the Coulomb energy of the uncorrelated electron crystal, for $N = 6$ particles. The superscript $2p$ on ^{2p}CFC indicates the vortex quantum number of composite fermions. The energy of the electron crystal for $L > 400$ is taken from Yannouleas and Landman [33]. The deviation of the exact energy from the electron crystal energy is also shown for $L \leq 145$; for larger angular momenta, where the exact energy is not available, we show an accurate approximation, $V_{\mathrm{CF}}^{(2)}$ (explained in the text), as an independent reference. For $2p > 6$, the number of vortices carried by composite fermions is shown in parentheses near the diamond. The energy difference per particle between the electron and the CF crystals is given in the inset, quoted in units of $e^2/\epsilon l_0$, where l_0 is the magnetic length and ϵ is the dielectric constant of the host semiconductor.

TABLE I. The last three columns give the overlaps of CF crystal (CFC), electron crystal (EC), and Laughlin's wave function with the exact ground state wave function at several filling factors ν. The overlap is defined as $|\langle\Psi^{\rm trial}|\Psi^{\rm exact}\rangle|^2/\langle\Psi^{\rm trial}|\Psi^{\rm trial}\rangle\langle\Psi^{\rm exact}|\Psi^{\rm exact}\rangle$. The second column gives D, the dimension of the basis space for $N = 6$ electrons, and L is the total angular momentum of the state.

ν (L)	D	CFC	EC	Laughlin
1/5 (75)	19 858	0.891	0.645	0.701
1/7 (105)	117 788	0.994	0.723	0.504
1/9 (135)	436 140	0.988	0.740	0.442

a better variational state than the electron crystal. Most significantly, the CF crystal is essentially the exact solution for $\nu \leq 1/7$ ($L \geq 105$). For $100 < L < 145$, the energy of the optimal CF crystal is approximately within 0.02% of the exact energy. Tables I and II show how Laughlin's liquid wave function, $\Psi^{\rm EC}$, and $\Psi^{\rm CFC}$ compare with the exact wave function for $\nu = 1/5$, $1/7$, and $1/9$. As indicated earlier, the liquid wave function worsens and $\Psi^{\rm EC}$ improves with decreasing ν, but neither is very good at small fillings. In contrast, $\Psi^{\rm CFC}$ is surprisingly close to the exact state. Its overlap with the exact wave function is $\sim$99% for $\nu = 1/7$ and $1/9$, while its energy deviates from the exact energy by 0.016% and 0.006%, respectively. It is worth noting the following: (i) The exact state is a linear combination of a large number of Slater determinant basis functions (see Table I), involving $D - 1$ parameters, and yet, a single CFC wave function captures its physics almost exactly. (ii) The CFC wave function for $\nu \leq 1/7$ is more accurate than Laughlin's wave function at $\nu = 1/3$, whose energy for $N = 6$ (in the disk geometry) is off by 0.15% and whose overlap with the exact state is 0.964, in spite of the fact that the dimension of the Fock space at $\nu = 1/3$ is much smaller ($D = 1206$). For larger L, the energy of the CFC is lower than $V_{\rm CF}^{(2)}$, with the possible exception of the last two points, where we may not have the optimal CFC.

Because every particle sees quantized vortices on every other particle, the formation of composite fermions implies a long range quantum coherence in the crystal phase. To get a feel for how the binding of vortices to electrons affects the interparticle correlations, we show in Fig. 2 the pair correlation function $g(x)$ for several candidate wave functions as well as the exact ground state for $\nu = 1/7$; $g(x)$ is the probability of finding a pair of particles at an arc distance x on a circle of radius R. (R is chosen to match the distance of a particle in the parent classical crystal from the center of the disk.) The result shows that the crystalline correlations are slightly weakened by the formation of composite fermions. It is perhaps counterintuitive that such an effect should lead to a lower energy even at very low fillings.

Of interest is the nature of the thermodynamic state, obtained in the limit $N \rightarrow \infty$ at a fixed filling factor. Finite size studies do not necessarily provide a reliable account of the thermodynamic state. For example, for $N = 6$ the CFC gives a better description of the $\nu = 1/5$ ground state than Laughlin's liquid wave function, even though the thermodynamic state here is known to be a liquid [7,8]. However, when an extremely precise and unambiguous description of the finite N state is obtained, as is the case at $\nu \leq 1/7$, we consider that to be a strong indication for the nature of the state in the thermodynamic limit. In any case, even though our finite N study cannot give the precise ν value where a transition from liquid to crystal takes place, it does make a compelling case that whenever the thermodynamic state is a crystal, it is a crystal of composite fermions, even in regions of the phase diagram far from the CF liquid.

The quantum character of the crystal is not fragile and ought to be observable at presently attainable temperatures, even at very small ν. The energy difference per particle, $V^{\rm CFC} - V^{\rm EC}$, shown in the inset of Fig. 1, gives a crude estimate for the temperature below which the quantum nature of the crystal should be robust to thermal fluctuations. The relevant temperatures appear to be well within the present experimental reach; for example, for

TABLE II. Interaction energies per particle for the exact ground state, the CF crystal (CFC), the electron crystal (EC), and Laughlin's wave function for six particles at several filling factors. The uncertainty in the last digit from Monte Carlo sampling is given in parentheses.

ν (L)	Exact	CFC	EC	Laughlin
1/5 (75)	2.2019	2.2042(5)	2.2196	2.2093(2)
1/7 (105)	1.8533	1.8536(2)	1.8622	1.8617(2)
1/9 (135)	1.6305	1.6306(1)	1.6361	1.6388(1)

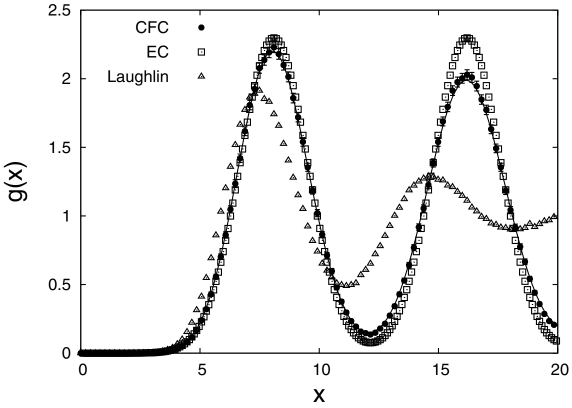

FIG. 2. The pair correlation functions for the CF crystal (solid circles), the electron crystal (empty squares), and Laughlin's wave function (empty triangles) on a circle of radius $R = 6.445 l_0$ for six particles at $\nu = 1/7$. The solid line shows the exact pair correlation function.

parameters appropriate for GaAs, the quantum crystal regime is estimated to be below ≈ 25 mK (at $B = 25$ T) even at $\nu = 1/33$. From the N dependence, we have estimated that the energy difference shown in the inset underestimates the thermodynamic energy difference by approximately a factor of 2. It is interesting to note that even as the energy difference between the CF and the electron crystals decreases as $\nu \to 0$, $2p$ continues to rise. Thus, CF flavors of up to very high $2p$ are predicted to occur in the crystal state. In the liquid phase, ^{2}CFs and ^{4}CFs have definitely been observed, and there is evidence also for ^{6}CFs and ^{8}CFs at relatively high temperatures [11].

Unlike in bosonic quantum crystals, the overlap between (uncorrelated) electron wave packets at neighboring sites is negligible in the filling factor region of interest (the overlap integral is 10^{-15} for $\nu = 1/9$). The quantum nature of the CF crystal owes its origin to the long range Coulomb interaction.

Given that the CF liquid behaves qualitatively differently from an electron liquid, one may ask in what ways the properties of the CF crystal are distinct from those of an electron crystal. We mention here a few examples where the CFC can provide natural explanations for certain experimental facts, although further work will be needed to make the connection with experiment more direct and to clarify other possible implications. The issue is obviously relevant to experiments that exhibit transitions between the liquid and crystal phases. Reentrant transitions between the FQHE liquid and an insulating state, thought to be a pinned crystal, have been observed [7,8] in going from $\nu = 2/5$ to $\nu = 1/5$. In the filling factor range $1/5 > \nu > 1/9$, the low temperature insulating state melts into a CF liquid upon raising temperature [11], as indicated by the appearance of FQHE-like structure. These observations become less baffling knowing that the crystal is itself made of composite fermions rather than electrons, thus requiring a less drastic reorganization of the state at the transition. Another result, perhaps puzzling for an electron crystal, is that the Hall resistance of the pinned crystal is close to the value it would have for a liquid [37]. If the current is carried by composite fermions instead, then the Hall voltage induced by the accompanying vortex current (the vortices effectively behave as magnetic flux quanta [32,38]), through an effective Faraday effect, is roughly consistent with the observation. (Zheng and Fertig [26] considered a similar mechanism for transport by correlated interstitial defects.) The unexpectedly small activation energy in the crystalline state, compared to theoretical predictions based on an electron crystal, as well as on its nonmonotonic filling factor dependence, has also been rationalized in terms of a CF crystal [27,28].

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