Self-Consistent Finite-Mode Approximations for the Hydrodynamics of an Incompressible Fluid on Nonrotating and Rotating Spheres

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Self-consistent finite-mode approximations for both Euler and Navier-Stokes equations for vorticity on a sphere are constructed and extended to the case of a rotating sphere, aiming at application to ocean and atmosphere modeling. In the absence of dissipation they preserve the specific Hamiltonian structure of hydrodynamics and have, at each level of approximation, an appropriate number of integrals of motion, which is not the case for standard schemes.

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The dynamics of the perfect incompressible fluid, if considered from the Eulerian viewpoint, provides a peculiar example of a noncanonical Hamiltonian system with an infinite number of integrals of motion, in particular, in two dimensions [1]. These integrals, called Casimir functions, originate from the invariance under relabeling of the Lagrangian particles and express, in the Eulerian language, the fundamental property of fluids to preserve the circulation along any liquid contour (Kelvin theorem). In geometric terms [2], incompressible inviscid 2D flows correspond to the area-preserving (symplectic) diffeomorphisms (one-to-one differentiable mappings of the domain of the flow onto itself, to be called sdiff below), which form an infinite-dimensional Lie group. The Hamiltonian structure is provided by the Lie-Poisson-Kirillov brackets defined in the space dual to the Lie algebra of this group. This Hamiltonian structure and all Casimir functions except for the quadratic one, the enstrophy, are lost if a straightforward truncation in the Fourier space is applied, e.g., for the purposes of numerical simulations. The problem of self-consistent hydrodynamical truncations thus arises. This problem is common for both Euler and Navier-Stokes equations, as even in the presence of viscosity an inaccurate discrete version of advection and pressure terms induces spurious effects which give accumulating errors with increasing integration time. For flows with periodic boundary conditions (flows on the torus T^2), this problem was solved by the so-called sine truncations [3,4]: a sequence of finite-mode approximations preserving the original symplectic structure and providing an appropriate number of Casimir functions for each size of truncation in the Fourier space. In addition, the Casimir functions tend to original ones when the truncation size tends to infinity. The idea of these self-consistent truncations was based on the interpretation of the 2D vorticity equation as an Euler system (see below) on the coalgebra of sdiffT^2 [2,4] on the one hand, and on the observation [5] that sdiffT^2 may be considered as a $N \to \infty$ limit of the finite-dimensional Lie algebras $su(N)$ on the other hand.

The sine truncations attracted considerable attention afterwards. A fast symplectic integrator for them was proposed in [6]; the difference in statistical behavior of the 2D turbulence simulated by standard versus sine-truncations methods was studied in [7,8], and recently in [9] in the context of the quasigeostrophic turbulence. Consistency, convergence, and stability of the sine truncations were proved in [10]. Below we will present self-consistent truncations on the sphere which are based on the same principle as sine truncations, and show how rotation may be accounted for.

Recall that for a N-dimensional Lie algebra $\mathcal{L}$ defined by the commutation relations

$$[L_i, L_j] = C_{ij}^k L_k, i, j, k = 1, 2, 3, \dots N, \tag{1}$$

among the *generators* L_i which form a basis in the linear space $\mathcal{L}$, a natural dynamical system, the *Euler system*, may be defined on $\mathcal{L}^*$, a space of 1 forms dual to $\mathcal{L}$:

$$\dot{\omega}_i = g^{lj} C_{ij}^k \omega_l \omega_k. \tag{2}$$

Here the dot denotes the time derivative, g is an arbitrary symmetric constant metric tensor, C_{ij}^k are the structure constants of $\mathcal{L}$, and summation over repeated indices is understood.

By antisymmetry of the structure constants the *energy* is conserved:

$$H = \frac{1}{2} g^{ij} \omega_i \omega_j, \qquad \dot{H} = 0. \tag{3}$$

Hamiltonian structure is provided by the Lie-Poisson-Kirillov bracket defined for any pair of functions $f(\omega)$, $g(\omega)$ on $\mathcal{L}^*$:

$$\{f, g\} = C_{ij}^k \omega_k \frac{\partial f}{\partial \omega_i} \frac{\partial g}{\partial \omega_j}. \tag{4}$$

Thus $\dot{\omega}_i = \{\omega_i, H\}$. The bracket (4) is degenerate, as there always exist the Casimir functions such that:

$$\{C(\omega), f\} \equiv 0, \qquad \forall f, \tag{5}$$

which are the integrals of motion.

The hydrodynamics on the sphere S^2 is a Euler system built from the structure constants of the Lie algebra of symplectic diffeomorphisms of the sphere sdiffS^2 and the metric tensor provided by the kinetic energy of the fluid. The vorticity field in spherical coordinates θ, ϕ is:

$$\omega(\theta,\phi) = \frac{\partial v_\phi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} + v_\phi \cot\theta, \tag{6}$$

and Euler and continuity equations are equivalent to the vorticity equation

$$\partial_t \omega + J(\Delta_S^{-1}\omega, \omega) = 0, \tag{7}$$

where Δ_S is the spherical Laplacian. The Jacobian in spherical coordinates is: $J(A,B) = \frac{1}{\sin\theta}(\partial_\theta A \partial_\phi B - \partial_\theta B \partial_\phi A)$. There is an infinite set of Casimir functions: any function of vorticity integrated over S^2 is conserved, in particular, any power of vorticity. The kinetic energy of the fluid is

$$H = -\frac{1}{2}\int_{S^2} \omega \Delta_S^{-1}\omega. \tag{8}$$

The standard spherical harmonics Y_{lm} provide an orthonormal basis on the sphere. Expanding the vorticity field in Y_{lm}:

$$\omega(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \omega^{lm} Y_{lm}(\theta,\phi), \tag{9}$$

and introducing quantities with lower indices via the relation: $\omega_{lm} = i\bar{\omega}^{lm} = i(-1)^m \omega^{l-m}$ allows to rewrite (7) in the form (2):

$$\dot{\omega}_{lm} = -\sum_{l'm'}\sum_{l''m''}[l'(l'+1)]^{-1}\gamma_{lm,l'm'}^{l''m''}(-1)^{m'}\omega_{l'-m'}\omega_{l''m''}, \tag{10}$$

where

$$\gamma_{lm,l'm'}^{l''m''} = i(-1)^{m''}\int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta Y_{l''-m''} J(Y_{lm}, Y_{l'm'}) \tag{11}$$

are the structure constants of sdiffS^2 which arise in the commutation relations:

$$iJ(Y_{lm}, Y_{l'm'}) = \gamma_{lm,l'm'}^{l''m''} Y_{l''m''}. \tag{12}$$

The metric tensor is provided by the decomposition of (8) in spherical harmonics and the use of the conjugation properties: $\bar{Y}_{lm} = (-1)^m Y_{l-m}$. In order to get a self-consistent truncation of (10) we introduce, following [5,11], the matrices $T_{lm}^{(N)}$ with matrix elements:

$$(T_{lm}^{(N)})_{m_1 m_2} = (-1)^{[(N-1)/2]-m_1}\sqrt{2l+1}\begin{pmatrix} \frac{N-1}{2} & l & \frac{N-1}{2} \\ -m_1 & m & m_2 \end{pmatrix}. \tag{13}$$

Here and below the notation (:::) and {:::} is used, respectively, for the $3j$ and $6j$ symbols of the angular momentum theory [12]. $T_{lm}^{(N)}$ for odd N have the same normalization properties as spherical harmonics:

$$Tr(T_{lm}^{(N)} T_{l'm'}^{(N)}) = (-1)^{N-1+m}\delta_{ll'}\delta_{m+m'0}, \tag{14}$$

and the following commutation relations:

$$[T_{lm}^{(N)}, T_{l'm'}^{(N)}] = f_{lm,l'm'}^{(N)l''m''} T_{l''m''}^{(N)} \tag{15}$$

with structure constants:

$$\begin{aligned} f_{lm,l'm'}^{(N)l''m''} &= [1-(-1)^{l+l'+l''}](-1)^{m''+1} \\ &\quad \times \sqrt{2l''+1}\sqrt{2l'+1}\sqrt{2l+1} \\ &\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}\begin{Bmatrix} l & l' & l'' \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix}. \end{aligned} \tag{16}$$

They form a representation of the algebra of unitary matrices $u(N)$ ($su(N)$ if the unit matrix $T_{00}^{(N)}$ is removed).

The matrices $T_{lm}^{(N)}$ and the spherical harmonics are intrinsically related. $T_{lm}^{(N)}$ are spherical tensors [12] built from the N-dimensional vector representation S_1, S_2, S_3 of the group of rotations:

$$[S_i, S_j] = i\epsilon_{ijk}S_k, \; i, j, k = 1, 2, 3. \tag{17}$$

Spherical harmonics may be expressed [12] in terms of harmonic polynomials of the Cartesian coordinates x_1, x_2, x_3 as:

$$r^l Y_{lm}(\theta,\phi) = \sum a_{i_1 \ldots i_l}^{(m)} x_{i_1}\cdots x_{i_l}, \; r^2 = x_1^2 + x_2^2 + x_3^2, \tag{18}$$

with $a_{i_1\ldots i_l}^{(m)}$ totally symmetric and traceless (for $l > 1$). Following [5,11], one gets $T_{lm}^{(N)}$ by replacing x_1, x_2, x_3 in the right-hand side of (18) by three matrices X_1, X_2, X_3 which are obtained from S_1, S_2, S_3 by renormalization with a factor $\frac{2}{\sqrt{N^2-1}}$. Thus X_i become commutative in the limit $N \to \infty$. The particular form (13) is recovered by using a special basis for S_i [11]. The structure constants $f_{lm,l'm'}^{(N)l''m''}$ are obtained from (13) by a direct computation using the definition of $6j$ symbols [12]. With a proper normalization they tend to $\gamma_{lm,l'm'}^{l''m''}$ as $N \to \infty$ [5].

Thus, the algebra sdiffS^2 may be recovered via the sequence of the $su(N)$ algebras (15) which may be used to construct the finite-mode analogs of the vorticity equation on the sphere:

$$\begin{aligned} \dot{\omega}_{lm} = -\sum_{l'=1}^{N-1}\sum_{l''=1}^{N-1}\sum_{m'=-l'}^{l'}\sum_{m''=-l''}^{l''}[l'(l' \\ +1)]^{-1} f_{lm,l'm'}^{(N)l''m''}(-1)^{m'}\omega_{l'-m'}\omega_{l''m''}. \end{aligned} \tag{19}$$

This system possesses $N-1$ Casimir functions which are obtained following [4] from the matrix invariants $\mathrm{Tr}[(\omega^{lm}T_{lm}^{(N)})^n]$, $n = 1, 2, \ldots, N-1$. They correspond to the integrated powers of vorticity, i.e., to the hydrodynamic Casimir functions.

Note that in the same way as ordinary Laplacian is built from the spherical harmonics:

$$\Delta_S \ldots = \frac{4\pi}{3}\bigg(J(Y_{10}, J(Y_{10}, \ldots)) - \frac{1}{2} J(\sqrt{2}Y_{11}, J(\sqrt{2}Y_{1-1}, \ldots)) - \frac{1}{2} J(\sqrt{2}Y_{1-1}, J(\sqrt{2}Y_{11}, \ldots))\bigg), \quad (20)$$

a discrete Laplacian

$$\Delta_N = \frac{N^2 - 1}{2}\bigg([X_3, [X_3, \ldots]] - \frac{1}{2}[X_+, [X_-, \ldots]] - \frac{1}{2} \times [X_-, [X_+, \ldots]]\bigg), \quad (21)$$

where $X_\pm = \mp(X_1 \pm iX_2)$ may be built from the operators X_i, $i = 1, 2, 3$ or, equivalently, from $T_{10}^{(N)}, T_{1\pm1}^{(N)}$ [11]. The matrices $T_{lm}^{(N)}$ are the eigenfunctions of the discrete Laplacian with the same eigenvalues $-l(l+1)$ as the eigenvalues of spherical harmonics with respect to Δ_S. This fact explains why the choice of metrics in (19) is the same as in (10) and, in addition, allows to get a discrete Navier-Stokes equation for vorticity on the sphere by adding the term $-\nu l(l+1)\omega_{lm}$ in the right-hand side of (19).

In order to describe the incompressible hydrodynamics on the sphere rotating with angular velocity Ω around its axis, the Coriolis force is to be added in the equations of motion. As usual in geophysical applications, the centrifugal force will be absorbed in the pressure and will not appear in the vorticity equation which may be rewritten in the form of the *potential vorticity* equation:

$$\partial_t q + J[\Delta_S^{-1}(q - 2\Omega \cos\theta), q] = 0, \quad (22)$$

where $q = \omega + 2\Omega \cos\theta$ is potential vorticity. If one notes that $\cos\theta = 2\sqrt{\frac{\pi}{3}}Y_{10}$, Eq. (22) in terms of decomposition in spherical harmonics takes the following form:

$$\dot{q}_{lm} = -\sum_{l'm'}\sum_{l''m''}(l'(l'+1)^{-1}\gamma_{lm,l'm'}^{l''m''}(-1)^{m'}) \times \bigg(q_{l'-m'} - 4\Omega\sqrt{\frac{\pi}{3}}\delta(l'-1)\delta(m')\bigg)q_{l''m''}. \quad (23)$$

Replacing $\gamma_{lm,l'm'}^{l''m''}$ by $f_{lm,l'm'}^{(N)l''m''}$ and limiting summation by $N-1$ gives consistent finite-mode truncations.

It should be emphasized that although the incompressible Euler or Navier-Stokes equations on the rotating sphere are, obviously, not realistic models for oceanic or atmospheric motions, the so-called quasigeostrophic equations (cf., e.g., [13]) for potential vorticity, which are obtained, e.g., from the rotating shallow-water equations by filtering the fast inertia-gravity waves, are. These equations for a single shallow-water layer have the same form as (22), up to a change of metrics, and thus allow for $su(N)$ truncations. Following the lines of [14] where periodic boundary conditions were used, consistent truncations may be also constructed for *multilayer* geostrophic models on the sphere, which are of extensive use in studies of climate dynamics (cf. [15]). Topographic effects may be easily included by adding corresponding terms to the potential vorticity (cf. [9] for the periodic case). It should be also mentioned that for flows on the rotating sphere, the proposed truncations provide a possibility to study finite-mode Rossby-waves dynamics in a consistent way [Rossby waves are solutions of linearized Eq. (22)].

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