# Stabilizing Unstable Steady States Using Multiple Delay Feedback Control

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Feedback control with different and independent delay times is introduced and shown to be an efficient method for stabilizing fixed points (equilibria) of dynamical systems. In comparison to other delay based chaos control methods multiple delay feedback control is superior for controlling steady states and works also for relatively large delay times (sometimes unavoidable in experiments due to system dead times). To demonstrate this approach for stabilizing unstable fixed points we present numerical simulations of Chua's circuit and a successful experimental application for stabilizing a chaotic frequency doubled Nd-doped yttrium aluminum garnet laser.

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*Introduction.*—For many physical experiments or technical applications it is desirable to stabilize steady states (equilibria) embedded in chaotic attractors. A typical example for this chaos control task is frequency doubled Nd-doped yttrium aluminum garnet (Nd:YAG) lasers where chaotic intensity fluctuations occur when the pump current exceeds some critical threshold [1,2]. This phenomenon is also called the *green problem* because frequency doubled Nd:YAG- lasers are often used as source of coherent green light in technical applications (e.g., holographic displays) where constant (cw) light output is required. Many attempts have been made to suppress the intensity fluctuations by means of chaos control methods [3]. A particular challenge provides (technologically important) *compact* frequency doubled Nd:YAG lasers that possess a strong tendency to become chaotic and whose fluctuations are in the MHz range, where fast control methods are required. Therefore, control algorithms that require $A/D$ conversion and numerical computations [such as the well-known Ott-Grebogi-Yorke method [4] ] are very difficult to apply to such fast systems and most researchers use methods that can be implemented using analog devices like occasional proportional feedback (OPF) [5,6], Pyragas's time delay autosynchronization (TDAS) [7,8], extended TDAS (ETDAS) [9,10], or N time delay autosynchronization (NTDAS) [11]. For stabilizing steady states Chang *et al.* [12] considered, in particular, the limit of vanishing delay time and the resulting control scheme was applied to electronic circuits [12] and to a laser system [13].

*Multiple delay feedback control.*—For frequency doubled Nd:YAG lasers and other systems it turned out that Pyragas's TDAS method (and its extensions) is very successful for stabilizing UPOs but less efficient to control unstable steady states. To overcome this limitation we suggest *multiple delay feedback control* (MDFC) where two (or more) delayed feedback signals with *different* delay times $\tau_i$ are used. In contrast to ETDAS or NTDAS these delay times are not integer multiples of each other and may enter independent control terms. For a general dynamical system given by some vector field $\mathbf{f}$ being controlled by two delayed signals the MDFC method reads

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + k_1\mathbf{g}_1[\mathbf{x}(t-\tau_1)] - k_2\mathbf{g}_1[\mathbf{x}(t)] + k_3\mathbf{g}_2[\mathbf{x}(t-\tau_2)] - k_4\mathbf{g}_2[\mathbf{x}(t)], \quad (1)$$

where the functions $\mathbf{g}_i$ represent the way the feedback is implemented and the parameters $k_i$ are used to vary the strengths and influence (gain) of the feedback terms. If an unstable fixed point of the given system [i.e., $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = 0$] is to be stabilized the gain factors have to fulfill the additional constraints $k_1 = k_2$ and $k_3 = k_4$. Otherwise a new steady state may be created that is not a fixed point of the vector field $\mathbf{f}$ (a case which is for some applications also of interest). If the delay times $\tau_i$ are *not* an integer multiple of each other (i.e., there exists no integer $m$ such that $\tau_1 = m\tau_2$ or $\tau_2 = m\tau_1$) the full control term will vanish only for steady states (fixed points) but *not* for (unstable) periodic orbits. This is the crucial difference between MDFC and (E)TDAS, because for the latter the control term may vanish already for periodic orbits (provided the delay time in the control term is an integer multiple of the period of the orbit to be stabilized).

To illustrate the MDFC approach we consider Chua's circuit [14] in its normalized form

$$\dot{x} = \alpha[x + f(x) - y], \qquad \dot{y} = -x + y - z + u(t), \qquad \dot{z} = \beta y + \gamma z \quad (2)$$

with nonlinearity

$$f(x) = m_1 x + \tfrac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)$$

and a MDFC control term

$$u(t) = k_1[y(t-\tau_1) - y(t)] + k_2[y(t-\tau_2) - y(t)]$$

depending and acting on the $y$ variable (that corresponds to a capacitor voltage in the circuit). For the parameter values $\alpha = 35.9$, $\beta = 75.7$, $\gamma = -1.203$, $m_0 = -0.855$, and $m_1 = -1.1$, a chaotic double scroll attractor (Fig. 1)

 

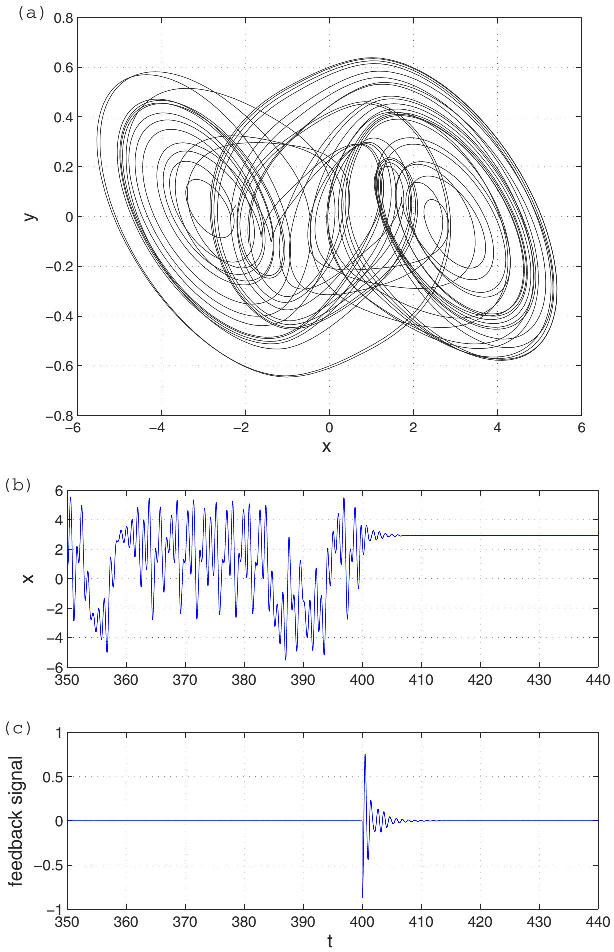


FIG. 1 (color online). (a) Chaotic attractor of the free running ($u = 0$) Chua's circuit (2). (b) $x$ variable and (c) feedback signal $u$ of Chua's circuit with control being activated at $t = 400$.

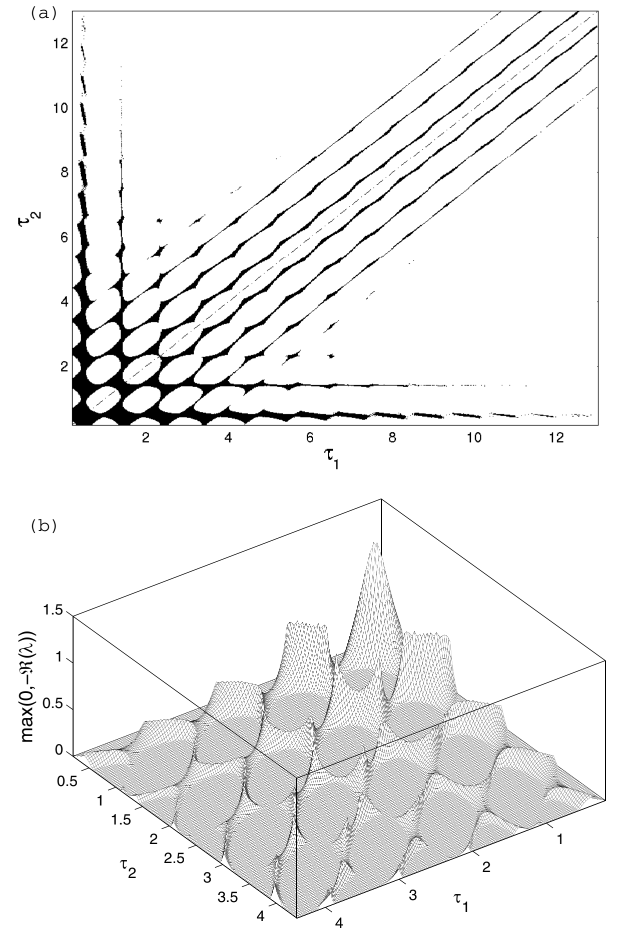


FIG. 2. (a) Stability diagram of the controlled Chua oscillator (2) for $k_1 = k_2 = 0.7$. Combinations of delay times $\tau_1$ and $\tau_2$ where MDFC successfully stabilizes the fixed point are denoted in black. The natural time scale (first maximum of the autocorrelation function) of the free running Chua oscillator equals $T = 0.94$. (b) Stability function $\max[0, -\mathcal{R}(\lambda)]$ vs delay times where $\mathcal{R}(\lambda)$ denotes the largest real part of the eigenvalues of the characteristic Eq. (3).

occurs for the noncontrolled circuit ($k_1 = k_2 = 0$). Embedded in this attractor are three fixed points [equilibrium points of the vector field (2)], one at the origin $(0, 0, 0)$ and the other two in the centers of the scrolls $[\pm(1 + \gamma/\beta)c, \pm\gamma c/\beta, \mp c]$ with $c = (m_0 - m_1)/(1 + m_1 + \gamma m_1/\beta)$. Figure 2(a) shows a stability diagram of the Chua example (2) where for fixed values of the coupling strengths $k_1 = 0.7 = k_2$ those regions in the $\tau_1$-$\tau_2$ parameter plane are marked in black where the stabilization of the equilibrium $[(1 + \gamma/\beta)c, \gamma c/\beta, -c]$ is successful. The dynamics of a (conventional) TDAS with a single delay term corresponds to the behavior along the diagonal [$\tau_1 = \tau_2$, denoted by a dash-dotted line in Fig. 2(a)]. As can be seen, no TDAS stabilization is possible for $\tau_1 = \tau_2 > 2.2$, whereas MDFC with $\tau_1 \neq \tau_2$ stabilizes the fixed point for much larger delay times [black lines parallel to the diagonal in Fig. 2(a)].

This stability diagram has been computed by linearizing the Chua ordinary differential equation (ODE) (2) at the fixed point $[(1 + \gamma/\beta)c, \gamma c/\beta, -c]$ and by computing the roots $\lambda$ of the transcendental characteristic equation

$$\det\begin{pmatrix} \lambda - \alpha(1 + \frac{df}{dx}) & \alpha & 0 \\ 1 & \lambda - 1 - g(\lambda) & 1 \\ 0 & -\beta & \lambda - \gamma \end{pmatrix} = 0 \quad (3)$$

with

$$\frac{df}{dx} = m_1 + \frac{1}{2}(m_0 - m_1)\left(\frac{x + 1}{|x + 1|} - \frac{x - 1}{|x - 1|}\right)$$

and

$$g(\lambda) = k_1(e^{-\tau_1\lambda} - 1) + k_2(e^{-\tau_2\lambda} - 1).$$

Since (2) is a delay differential equation this characteristic equation possesses an infinite number of roots. For describing the stability properties of the fixed point possible roots with positive real parts are most important because they correspond to unstable directions. To locate these eigenvalues is a nontrivial task [15]. It is, however, facilitated by the fact that for this type of linear delay ODEs only a finite number of roots with real parts larger than a given constant occur [16]. To find these roots we first use a grid in the relevant part of the complex plane to detect $\lambda$ values for which both the real part and the imaginary part of the complex function (3) vanish (or are very close to zero). These candidates for eigenvalues are then used as initial values of a damped Newton's algorithm to compute their exact location. To analyze the dynamical features of our control scheme we consider the eigenvalue with the largest real part $\mathcal{R}(\lambda)$. If this real part is positive the control scheme fails; if it is negative MDFC works and is the more robust the larger the magnitude of the negative real part is. To visualize these stability properties $\max[0, -\mathcal{R}(\lambda)]$ is plotted vs the delay times $\tau_1$ and $\tau_2$ in Fig. 2(b). The higher the peaks the more stable is the corresponding fixed point stabilization. Except for $\tau_i < 1$ different delay times ($\tau_1 \neq \tau_2$) provide higher stability (and thus more robustness with respect to noise) than conventional TDAS ($\tau_1 = \tau_2$).

MDFC is also more efficient than ETDAS [9,10] as illustrated for Chua's circuit in Fig. 3 which has been computed for an ETDAS control signal

$$u(t) = k[y(t - \tau) - y(t)] + Ru(t - \tau). \qquad (4)$$

Including multiple delays [$R = 0.7$, Fig. 3(b)] extends the regions in the $k$-$\tau$ plane where the fixed point is successfully stabilized (black areas in Fig. 3) compared to TDAS control [$R = 0$, Fig. 3(a)], but for delay times $\tau > 9$ ETDAS also fails. For systems with dead times (control loop latencies) MDFC with different delay times is thus superior to TDAS and ETDAS.

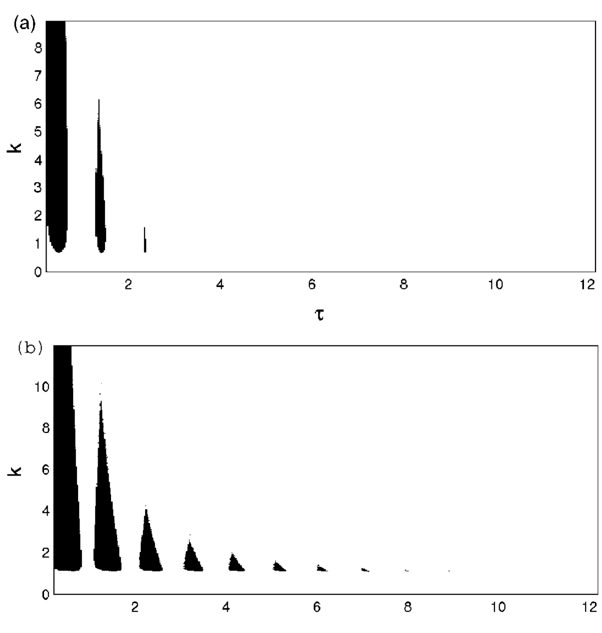


FIG. 3. Stability diagram for ETDAS where the control signal (4) is applied to Chua's circuit (2). Parameter combinations $(\tau, k)$ for which succesful fixed point stabilization is achieved are marked in black. (a) $R = 0$ (TDAS). (b) $R = 0.7$ (ETDAS).

*Laser stabilization.*—Since the first investigation of chaotic intensity fluctuations of frequency doubled Nd:YAG lasers by Baer [1] many attempts have been made to cope with this type of dynamical instability. Optical counter measures include adding a quarter-wave plate to the cavity [17,18], rotating the KTP crystal [19,20], or using an L-shaped cavity [21]. On the other hand, many authors investigated the chaotic dynamics of frequency doubled Nd:YAG lasers and applied different chaos control methods. Roy *et al.* [22] succeeded in experimentally stabilizing unstable periodic orbits (UPOs) using OPF. Gills *et al.* [23] combined this approach with a tracking method to extend the stability region. Numerically, OPF control of UPOs was demonstrated by Colet *et al.* [24] using a multimode laser model and by Carr and Schwartz [25] who stabilized fixed points with a modified OPF method. Alternatively, a conventional proportional differential (PD) controller based on two orthogonally polarized infrared intensities has been used by Pyragas *et al.* [26,27] for fixed point stabilization in numerical simulations of frequency doubled Nd:YAG lasers. However, OPF as well as PD control was successful for low pump currents or in numerical simulations, only. At higher pump currents more complex so-called type-II chaos [28,29] occurs where modes in two perpendicular polarization directions are active. This chaotic dynamics turned out to be difficult to be tamed and the most detailed experimental investigation until now was presented by Schenck zu Schweinsberg and Dressler [30] using a PD controller with two infrared intensities as input signals. Their experimental laser system showed chaotic oscillations of type II at about 30 KHz that they were able to stabilize at medium pump rates.

Motivated by the severe limitations of all previously suggested and reported attempts for solving the green problem using control methods we experimentally applied MDFC to suppress chaotic intensity fluctuations of a frequency doubled Nd:YAG laser at higher pump rates.

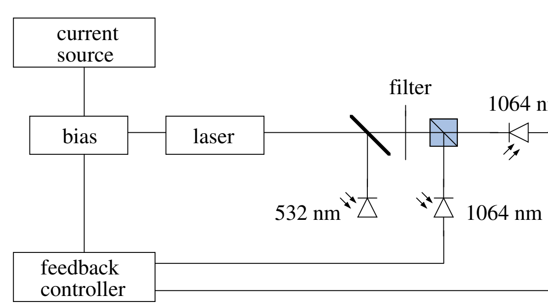


FIG. 4 (color online). Experimental setup for stabilizing chaotic (type-II) intensity fluctuations of a green (532 nm) frequency doubled Nd:YAG laser using MDFC. Infrared (1064 nm) intensities in two orthogonal polarization directions are measured (ac components) and fed back to modulate the pump current.

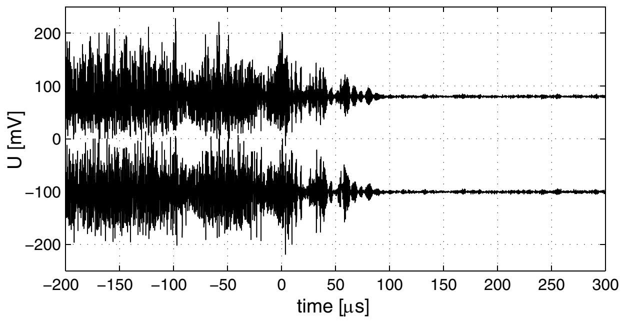


FIG. 5. Experimental suppression of chaotic intensity fluctuations of a frequency doubled Nd:YAG laser using MDFC. Shown are both orthogonally polarized ac infrared signals $S_x$ (upper trace) and $S_y$ (lower trace) with control being switched on at $t = 0$.

Because of its compact design (housing of pump diode: 6 cm; housing of Nd:YAG and KTP crystal: 7 mm) this laser oscillates chaotically (type II) with dominant frequencies between 1 and 1.5 MHz. The power of the emitted (chaotic) infrared and green light equals 20 mW and 2 mW, respectively. Figure 4 shows the experimental setup where two ac-coupled infrared (1064 nm) orthogonally polarized intensities $S_x$ and $S_y$ of the emitted light are measured. These signals are delayed in an analog device (all-pass filters operating as Bessel filters) and are fed back to the pump current using a bias $T$.

The complete control loop may be written as

$$u(t) = a_x S_x(t) - b_x S_x(t - \tau_x) + a_y S_y(t) - b_y S_y(t - \tau_y) \tag{5}$$

and depends on six parameters ($a_x$, $b_x$, $a_y$, $b_y$, $\tau_x$, $\tau_y$) that were experimentally chosen to achieve stabilization of the steady state. Typical delay times are $\tau_1 = 0.6$ $\mu$s and $\tau_2 = 2.8$ $\mu$s. Figure 5 shows the onset of stabilization at a pump power which was 3 times larger than the laser threshold with typically short transients. The stabilized green light has a power of 1.5 mW and consists of three active modes (in contrast to four to five modes observed with the free running chaotic laser). Attempts to stabilize the laser at this intensity level with a PD controller or single delay TDAS failed.

Feedback control with different delay times has also successfully been applied to stabilize steady states of chaotic electronic circuits and various other dynamical systems. In all these cases using MDFC the stability region of steady states could be drastically enlarged. Open questions for future research are possible further improvements using more than two delay times, applications to high dimensional systems, and progress in the theoretical understanding of dynamical features of MDFC.

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