Bose-Einstein Condensation and Strong-Correlation Behavior of Phonons in Ion Traps

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We show that the dynamics of phonons in a set of trapped ions interacting with lasers is described by a Bose-Hubbard model whose parameters can be externally adjusted. We investigate the possibility of observing several quantum many-body phenomena, including Bose-Einstein condensation as well as a superfluid-Mott insulator quantum phase transition.

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Systems of ultracold bosons present a rich variety of fascinating phenomena, like Bose-Einstein condensation (BEC) [1], or the superfluid-Mott insulator (SI) quantum phase transition [2]. At present, there exist very few physical systems in which these effects can be observed. Atomic gases constitute a unique system, since their physical parameters can be adjusted using external fields, which has enabled the observation of BEC [3] or the SI transitions [4]. In this Letter we show that phonons in a crystal of trapped ions interacting with lasers provide us with another system where all these phenomena can be observed in a very clean way. As for neutral atoms, the physical parameters describing the phonon dynamics can be adjusted using lasers. Furthermore, individual addressing yields new possibilities for investigating novel physical situations.

In our setup, the phonons are associated with the motion of the ions. Coulomb interaction induces the transmission of phonons from one ion to another, whereas anharmonicities in the trapping potentials give rise to an effective phonon-phonon interaction. Thus, an ion crystal is analogous to an optical lattice [5], whereby the ions play the role of lattice sites and the phonons that of the atoms. An important feature is that, due to energy conservation, phonons cannot be created or annihilated. This is in contrast with usual solid state systems, where phonons are subjected to processes that do not conserve their number thus preventing them from reaching, e.g., BEC. Furthermore, the theoretical [6,7] and experimental progress [8–10] in the field of trapped ion quantum computation can be exploited in the present context to gain access to physical observables which are not reachable in other systems.

Let us consider N trapped ions confined by external electric potentials and which move around their equilibrium positions. The corresponding Hamiltonian is $H = K + V_0 + V_{\rm Coul}$, where K describes the kinetic energy, V_0 the trapping potential, and $V_{\rm Coul}$ is the Coulomb interaction between ions. We will assume that: (i) the motion of the ions along one particular direction, say $\mathbf{x}$, is decoupled from the motion along the other directions; (ii) the trapping potential along $\mathbf{x}$ for each ion is practically harmonic with frequency ω, i.e., it is given by $\frac{1}{2}m\omega^2 x_i^2$, where x_i denotes the operator corresponding to the displacement of the ith ion; (iii) the displacements around the equilibrium position are much smaller than the distances between ions; (iv) the Coulomb energy is small compared to the potential energy, i.e., $\beta := e^2/(d_0^3 m\omega^2) \ll 1$, where d_0 denotes the average separation between ions. The first requirement (i) allows us to ignore the motional state of the ions along the $\mathbf{y}$ and $\mathbf{z}$ directions. Condition (ii) allows us to associate phonons to each of the ions in the usual way [7]: if the vibrational state of the ith ion is given by the nth (Fock) excitation state of the corresponding harmonic potential we will say that the ion has n phonons and denote the corresponding state by $|n\rangle$. The position operator of the ion can be then written in terms of creation and annihilation operators for the phonons, i.e., $x_i \propto (a_i + a_i^\dagger)$. Condition (iii) allows us to express the Coulomb interaction between ions i and j as $(e^2/d_{i,j}^3)x_i x_j$, where $d_{i,j}$ is the distance between the ions. It is clear that this term will induce hopping of phonons between the ions since it contains terms of the form $a_i^\dagger a_j$. On the other hand, condition (iv) imposes that the phonon number is conserved, since the terms of the form $a_i a_j$ would decrease the energy by 2ω, something which cannot be compensated by the Coulomb interaction. Finally, the anharmonicities of the trapping potential will be, in lowest order, described by terms of the form x_i^3 or x_i^4. Again, only energy-conserving terms will be important and thus only those proportional to $a_i^\dagger a_i$ or $a_i^{\dagger 2} a_i^2$ will survive. The first one will add some small correction to the trapping frequency, whereas the second term can be associated to an effective phonon-phonon interaction.

Thus, we have shown that the dynamics of the phonons in an ion crystal will contain hopping terms, as well as on-site phonon-phonon interactions, and therefore they will be described by a Bose-Hubbard model (BHM). Typically, the trapping anharmonicities will be very small. However, they can be enhanced by using off-resonant lasers. For instance, one may induce repulsive (attractive) phonon-phonon interactions by placing the ions near the maximum (minimum) of a standing wave, something which will induce an ac-Stark shift $\propto \cos(kx_i)^2 \approx 1 \mp (kx_i)^2 \pm (1/3)(kx_i)^4$, where k is the wave vector of the laser.

There are several physical setups which realize a BHM as explained above. In the following we will concentrate in the simplest one, which consists of ions in a linear trap. Let us emphasize, however, that with ions in microtraps [11] or in Penning traps [12] one can realize higher dimensional situations.

In a linear trap, ions are arranged in a Coulomb chain. Phonons moving along the chain cannot be used in the way we described above since for them $\beta \gtrsim 1$ [13,14]. However, transverse phonons corresponding to the radial modes fulfill $\beta \ll 1$ and thus are perfectly suited for our purposes. The radial phonon dispersion relation has a large gap of the order of ω, giving rise to the phonon-number conservation, and has a bandwidth of the order of $\beta\omega$. Therefore, we take $\mathbf{x}$ as one of the transverse directions and $\mathbf{z}$ the trap axis. The Hamiltonian of a chain with N ions is

$$\begin{aligned} V_0 &= \frac{1}{2} m \sum_{i=1}^{N} (\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2), \\ V_{\mathrm{Coul}} &= \sum_{i>j}^{N} \frac{e^2}{\sqrt{(z_i - z_j)^2 + (x_i - x_j)^2 + (y_i - y_j)^2}}, \end{aligned} \tag{1}$$

where ω_α, $\alpha = x, y, z$, are the trapping frequencies in each direction, and we define β_α as the ratios between Coulomb and trapping energy. If $\omega_{x,y} \gg \omega_z$ the ions form a chain along the $\mathbf{z}$ axis and occupy equilibrium positions z_i^0. Phonons in the $\mathbf{x}$ direction can be described approximately by

$$\begin{aligned} H_{x0} &= \sum_{i=1}^{N} (\omega_x + \omega_{x,i}) a_i^\dagger a_i + \sum_{j>i}^{N} t_{i,j} (a_i^\dagger a_j + a_i a_j^\dagger), \\ \omega_{x,i} &= -\frac{1}{2} \sum_{j' \neq i}^{N} \frac{e^2/(m\omega_x^2)}{|z_i^0 - z_{j'}^0|^3} \omega_x, \; t_{i,j} = \frac{1}{2} \frac{e^2/(m\omega_x^2)}{|z_i^0 - z_j^0|^3} \omega_x. \end{aligned} \tag{2}$$

$\omega_{x,i}$ are spatial dependent shifts of the trapping frequency, and $t_{i,j}$ are hopping energies. Note that both of them are of the order of $\beta_x \omega_x$. The approximations that lead to (2) are (i) In V_{Coul}, we keep only second order terms in the displacements of the ions. Higher order terms are of the form x^4, x^2y^2, zx^2. They can be neglected assuming that $x_0/d_0, z_0/d_0 \ll 1$, with x_0, z_0 the size of an individual ion's wave packet. x_0 can be estimated by the size of the ground state in the radial trapping frequency and for typical parameters ($d_0 = 5$ μm, $\omega_x =$ 10 MHz, also used below) we have $x_0/d_0 \approx 10^{-3}$. In the case of z_0, one has to consider the collective nature of the axial modes, because $\beta_z \gg 1$ if $N \gg 1$. If we consider axial modes at a finite T, z_0 is given by the thermal fluctuations of the position of the ion. In the limit $N \gg 1$, we can estimate $z_0^2 \approx \hbar/2m\omega_z\sqrt{\beta_z \log\beta_z}(k_B T/\hbar\omega_z)$, which means that $z_0/d_0 \approx 10^{-3} k_B T/\hbar\omega_z$, with $\omega_z =$ 100 kHz, $N = 100$ (see [15]). (ii) We consider $\beta_x \ll 1$, and neglect phonon nonconserving terms in the couplings of the form $x_i x_j \propto (a_i^\dagger + a_i)(a_j^\dagger + a_j)$.

We also include the effect of a standing wave in H_x, such that a repulsive phonon-phonon interaction is induced [7]: $H_{sw} = F \sum_{i=1}^{N} |0\rangle_i\langle 0| \cos^2(kx_i)$, where $|0\rangle_i$ is the internal ground state. In the following we will assume that ions stay always in $|0\rangle_i$, and expand the standing wave in the Lamb-Dicke parameter, $\eta = kx_0/\hbar\omega_x$:

$$H_{sw} = F \sum_{i=1}^{N} [1 + \eta^2 (a_i + a_i^\dagger)^2 + \frac{1}{3} \eta^4 (a_i + a_i^\dagger)^4 + \mathcal{O}(\eta^6)]. \tag{3}$$

The fourth order contribution contains a Hubbard interaction, $U a_i^{\dagger 2} a_i^2$, with $U = 2F\eta^4$ (note that $U < 0$ if the ions are placed at the minimum). The other terms in Eq. (3) are (i) phonon conserving terms that just give corrections to the trapping frequency; (ii) phonon nonconserving terms, that rotate with frequency ω_x. The nonconserving contributions can be adiabatically eliminated if $F\eta^2/\omega_x \ll 1$. For example, in the case of the second order terms, $F\eta^2(a_i^2 + a_i^{\dagger 2})$, a perturbative calculation shows that they only give harmonic corrections of the form $[(F\eta^2)^2/\omega_x](-2a_i^\dagger a_i - 1) + \mathcal{O}(F\eta^2)^3/\omega_x^2$. Thus, the contributions from phonon-number nonconserving terms either give corrections to the trapping frequency or can be neglected when compared to U.

The final Hamiltonian takes the form of a BHM:

$$H_x = H_{x0} + \sum_{i=1}^{N} U a_i^{\dagger 2} a_i^2, \tag{4}$$

where we include in H_{x0} the corrections from the standing wave. Note that as long as the number of phonons is conserved, ω_x in H_{x0} is a global chemical potential that does not play any role in the description of the system.

We discuss now the properties of the solutions of the noninteracting Hamiltonian, H_{x0}. A quite unexpected result is that the Coulomb interaction induces the confinement of the radial phonons. This is due to the fact that the distance between ions is larger at the sides than at the center of the chain, and is well described by a quadratic dependence on the position of the ions. Thus, the corrections $\omega_{x,i}$ in Eq. (2) are smaller for the ions placed at the center, in such a way that the radial phonon field is confined (Fig. 1, left). The harmonic phonon confinement can be estimated by means of Eq. (2) in the limit $N \gg 1$. In this case, the distance between ions at site i satisfies [13]:

$$\frac{1}{[d(i')/d_0]^3} \approx \alpha - \gamma\left(\frac{i'}{N}\right)^2, \qquad \alpha = 3.4, \qquad \gamma = 18, \tag{5}$$

where $i' = i - N/2$. One can use Eq. (5) to describe qualitatively the dependence of $\omega_{x,i}$ with the position. We include only Coulomb interaction between nearest neighbors in order to get analytical results. The spatial dependent part of the noninteracting boson Hamiltonian is given, in this approximation, by

$$H_{x0}/(\beta_x\omega_x) = \sum_{i=1}^{N}\frac{1}{2}\gamma\left(\frac{i'}{N}\right)^2 a_i^\dagger a_i + \frac{1}{2}\left[\alpha - \gamma\left(\frac{i'}{N}\right)^2\right] \times (a_i^\dagger a_{i+1} + \text{h.c.}). \quad (6)$$

In the limit of many ions and low energies, the continuum limit in this expression describes a one-dimensional system of bosons trapped by the frequency $\omega_c \approx (8/N)\beta_x\omega_x$. The lowest collective modes in the exact spectrum show a linear dispersion that is well described by our estimation for ω_c (see inset of Fig. 1, left). Thus, radial phonons in linear Paul traps are confined by an approximate harmonic potential (Fig. 1, right).

Our ideas lead to the following two proposals of experiments with linear Paul traps:

(i) Superfluid-Mott insulator transition and creation of a superfluid phonon state by adiabatic evolution.—Hamiltonian (4) describes a BHM with the peculiarity that hopping terms are positive and long range. However, we can understand the properties of our system by means of the better known model with nearest-neighbor hopping only [2,16]. Let us consider $t = (1/2)\beta_x\hbar\omega_x$, the characteristic hopping energy. If the total number of phonons N_{ph} is commensurate with the number of ions N, then, for $U \gg t$, the ground state of (4) is a Mott insulator, well described by a product of Fock states with N_{ph}/N phonons in each ion (note that phonon confinement, $\hbar\omega_c$ is also of order t, so that condition $U \gg t$ ensures a uniform phonon density). On the other hand, the ground state for $U \ll t$ is a superfluid with all the phonons in the lowest energy level. In Fig. 2, we present the results of an exact numerical diagonalization of the complete phonon Hamiltonian (that is, including also the phonon-number nonconserving terms) for the case $N_{\text{ph}}/N = 1$. The transition from the superfluid to the Mott insulator, is evident in the evolution of the phonon density as a function of the interaction U.

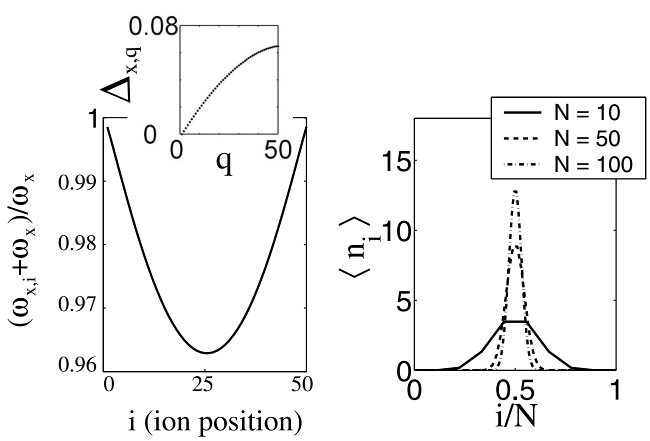

FIG. 1. Left: Phonon trapping potential (2) for the radial modes of a Coulomb chain with $N = 50$ ions, and $\beta_x = 10^{-2}$, as a function of the ion position along the chain. Left, inset: Spectrum of the radial collective modes that diagonalize H_{x0}. We plot the energy of the modes relative to the minimum phonon energy, $\Delta_{x,q} = (\Omega_{x,q} - \Omega_{x,0})/\omega_x$, where $\Omega_{x,q}$ is the energy of the collective modes, and q is the mode number. Right: Mean phonon-number $\langle n_i\rangle = \langle a_i^\dagger a_i\rangle$ along Coulomb chains with different number of ions, N, in the state with $N_{\text{ph}} = N$ phonons in the radial lowest mode. The width of the wave function in units of N, is $\propto 1/\sqrt{N}$, in accord with the scaling for the phonon trapping frequency, $\omega_c \propto 1/N$.

The properties of the BHM allows us to propose an experimental sequence that would lead to the observation of the SI quantum phase transition: (1) The ion chain is cooled to the state with zero radial phonons by laser cooling. (2) Starting with a value $U \gg t$, the eigenstates of the system are well described by Fock states localized at each ion. The ground state of the phonon system can be created by means of sequences of blue/red sideband transitions, in a method that has been successfully implemented with single trapped ions (see [8]). (3) The value of U is varied adiabatically down to a given value U_f, in such a way that the system remains in the ground state. At a given critical value $U_f \approx t$, the system undergoes a transition to a phonon superfluid. (4) Properties of the phonon ground state are measured by the method explained below.

Phonons cannot be directly detected, but the measurement of the phonon state can be accomplished by the coupling to a given internal transition. One can apply, for example, a red sideband pulse with intensity g, for a short time t, and measure the photoluminescence from each ion (see [17] for the setup and requirements). This experiment can be repeated many times to obtain the averaged photoluminescence, which corresponds to the transition probability to the excited internal state, $P_\uparrow(t) = \sum_n[\sin(\sqrt{n}gt)]^2 P(n) \approx \sum_n ng^2t^2P(n)$, where $P(n)$ is the probability of having n phonons. In this way the SI transition is evidenced in the variations of the phonon density along the chain (Fig. 2). Furthermore, detection of $P_\uparrow(t)$ for longer times would allow us to measure $P(n)$ [8], and detect the phonon-number fluctuations in the superfluid phase, even without resolving the ions individually. Finally, quantum tomography would allow us to get the whole phonon density matrix [10].

(ii) Bose-Einstein condensation by evaporative laser cooling.—We propose an experiment that is akin to the usual BEC of cold atoms in harmonic traps. First, we note that techniques for cooling of trapped ions, like laser cooling [10,18] can only be used to destroy phonons. The existence of the trapping phonon potential in ion traps allows us to propose the combination of laser cooling with the idea of evaporative cooling. A possible experimental sequence would be as follows: (1) Start with a Coulomb chain after usual Doppler cooling, that is, a chain with a given number of phonons per site, and induce a small phonon-phonon interaction $U \ll t$, so that the system remains in the weak interacting regime. (2) Apply laser cooling at the sides of the Coulomb chain, in such a way that the higher energy phonons on the top of the confinement potential are destroyed (evaporated).

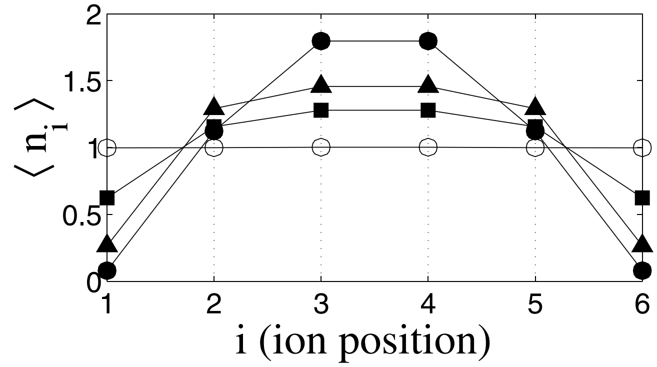

FIG. 2. Mean phonon number at each ion in the ground state of a Coulomb chain with $N = 6$, $N_{\mathrm{ph}} = 6$. $\beta_x = 0.01$, so that the nearest-neighbor hopping terms are $t \approx 510^{-3}\omega_x$. Black circles: phonon density without standing wave ($U = 0$), which shows the confinement due to the phonon trapping potential. Empty circles: Mott phase when a standing wave is applied with $F\eta^2 = 0.1\omega_x$, $\eta^2 = 0.1$, and $U \approx 0.02\omega_x > t$. Squares: $U = 0.01\omega_x$, Triangles: $U = 5 \times 10^{-3}\omega_x$.

(3) The interaction U induces collisions that thermalize the phonons to a lower temperature. Several cycles of laser cooling/thermalization can be applied until the system is cooled below the temperature for condensation. Detection of the BEC can be accomplished along the same lines exposed above for the case of the BHM. In the 1D case considered here, BEC would not be possible at finite T, in the thermodynamical limit ($N \to \infty$). In finite systems, however, there exists a finite temperature at which the occupations of the excited states saturate, and phonons condense in the lowest mode [22].

We have shown that phonons in a system of trapped ions can be manipulated in such a way that they undergo BEC, or a SI transition. The main ingredients of our proposal are (1) Phonons can have a large energy gap that suppresses processes that do no conserve the number of phonons. (2) Phonon-phonon interactions (anharmonicities) can be induced by placing the ions in a standing wave. (3) In the particular case of ions in a linear trap, radial phonons would be suitable for this proposal, and Coulomb interaction provides us with an approximately harmonic phonon confinement.

In this work we have exposed only a few applications of this idea, but phonons in trapped ions can be used to study quantum phases with a degree of controllability that is not possible with cold neutral atoms. Individual addressing would allow us to design Hubbard Hamiltonians with site dependent interactions. Different directions of the radial modes, or different internal states, would play the role of effective spins [19] for the phonons. On the other hand, one can also reach the regime dominated by the repulsive interaction and create, thus, a Tonks-Girardeau gas of phonons [20,21] in a Coulomb chain. In a very promising approach, 2D systems of arrays of microtraps [11], or ions in Penning traps [12], can be considered, because phonons transverse to the crystal plane satisfy the conditions required by our proposal.

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[1] A. Einstein, Sitzber. Kgl. Preuss. Akad. Wiss. **261** (1924); **3** (1925); S. N. Bose, Z. Phys. **26**, 178 (1924).
[2] M. P. A. Fisher *et al.*, Phys. Rev. B **40**, 546 (1989).
[3] M. H. Anderson *et al.* Science **269**, 198 (1995); K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995).
[4] M. Greiner *et al.*, Nature (London) **415**, 39 (2001).
[5] D. Jaksch *et al.*, Phys. Rev. Lett. **81**, 3108 (1998).
[6] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[7] J. I. Cirac *et al.*, Adv. At. Mol. Opt. Phys. **37**, 237 (1996).
[8] D. M. Meekhof *et al.*, Phys. Rev. Lett. **76**, 1796 (1996).
[9] D. J. Wineland *et al.*, J. Res. Natl. Inst. Stand. Technol. **103**, 259 (1998); D. Leibfried *et al.*, Phys. Rev. Lett. **89**, 247901 (2002); S. Gulde *et al.*, Nature (London) **421**, 48 (2003); F. Schmidt-Kaler *et al.*, Nature (London) **422**, 408 (2003).
[10] D. Leibfried *et al.*, Rev. Mod. Phys. **75**, 281 (2003).
[11] Ralph G. de Voe, Phys. Rev. A **65**, 063407 (2002).
[12] W. M. Itano *et al.*, Science **279**, 686 (1998); T. B. Mitchell *et al.*, Science **282**, 1290 (1998).
[13] D. H. E. Dubin and T. M. O'Neil, Rev. Mod. Phys. **71**, 87 (1999).
[14] A. Steane, Appl. Phys. B **64**, 623 (1997); D. F. V. James, Appl. Phys. B **66**, 181 (1998).
[15] Note that higher order contributions in the expansion of the Coulomb energy of the form x^2y^2, x^2z, contain terms that do not conserve the number of phonons in the x-direction. However, these terms can be eliminated in a rotating wave approximation under the condition that the trapping frequencies in each direction are different: $\omega_x \neq \omega_y \neq \omega_z$.
[16] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[17] F. Diedrich *et al.*, Phys. Rev. Lett. **62**, 403 (1989).
[18] C. Monroe *et al.*, Phys. Rev. Lett. **75**, 4011 (1995); Ch. Roos *et al.* Phys. Rev. Lett. **83**, 4713 (1999).
[19] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).
[20] M. Girardeau, J. Math. Phys. (N.Y.) **1**, 516 (1960).
[21] B. Paredes *et al.*, Nature (London) **429**, 277 (2004).
[22] The situation here is different from that of nonconserved bosons like photons, where the chemical potential is zero, which leads for example to the blackbody thermal distribution.