Transport Statistics of Bistable Systems

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We consider the transport statistics of classical bistable systems driven by noise. The stochastic path integral formalism is used to investigate the dynamics and distribution of transmitted charge. Switching rates between the two stable states are found from an instanton calculation, leading to an effective two-state system on a long time scale. In the bistable current range, the telegraph noise dominates the distribution, whose logarithm is found to be universally described by a tilted ellipse.

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Bistable systems are ubiquitous in nature. They occur in solid state [1,2], magnetic [3], and biological systems [4]. The behavior of bistable systems may be characterized by random switching between two stable states, induced by either thermal fluctuations or nonequilibrium noise. The presence of bistability is usually deduced by measuring the current passing through the system. The switching of the internal state of the system alters the resistance of the sample, causing the measured current to randomly switch between two distinct average values. The noise of this switching current is called random telegraph noise [5], a process studied by Machlup in the 1950s. An important feature of this process is that its zero-frequency noise power diverges as the switching rates tend to zero.

Fluctuations in bistable electronic devices, such as tunnel diodes, were theoretically investigated by Landauer [6] and later by Hänggi and Thomas [7]. Since then, bistable fluctuations have been observed in microstructures by different experimental groups [8]. The switching times were found to be well fit by an exponential of the applied bias. More recently, similar physics was discovered in mesoscopic resonant tunneling wells near an instability [9]. In this Letter, we examine the properties of bistable systems from the novel point of view of transport statistics. This approach gives not only the average current, and zero-frequency noise [10], but all higher current cumulants as well (also called full counting statistics in mesoscopic physics [11]). For many systems, high order cumulants may be computed via the cascade method [12], which uses the idea that the noise itself has noise [12–14] and that feedback effects of the lower order cumulants may be taken into account in a perturbative manner [15,16]. In contrast, for unstable systems, nonlinearities enter the transport in an essentially *nonperturbative* fashion. Here the situation is reversed: All higher cumulants of the noise sources must be accounted for to correctly find the switching rates which govern even the average current.

In order to successfully describe the physics of such an instability, we apply the stochastic path integral (SPI), a nonperturbative method that was recently introduced [15,16] to evaluate the statistical fluctuations in a classical stochastic network. The network is composed of nodes containing effectively continuous charge representing any conserved quantity. The charge is transferred through connectors whose isolated transport properties are given. In order to emphasize the universality of the results and focus on the essential physics, we use an abstract description which can represent many of the physical systems mentioned thus far, such as the tunneling diode, or more general *S*- or *Z*-type instabilities [2].

We now briefly describe our main results. We let the average left current have a common *S*-type nonlinear dependence on the charge on the nodes (see Fig. 1), and we demonstrate bistability. The connector noise induces switching between the stable states with the rates $\Gamma_{1,2}$, found via an instanton calculation, Eq. (9), to be exponentially small with the instability charge scale. On the time scale of order $\Gamma_{1,2}^{-1}$ or longer, the system can be viewed as being effectively a two-state system. After this point, results (12)–(15) hold independently of how the rates $\Gamma_{1,2}$ are obtained. The requirement of continuous charge may be relaxed, and the results apply to any telegraph process [1–4] with the rates as input parameters. The distribution of the transmitted charge is dominated by one of the stable

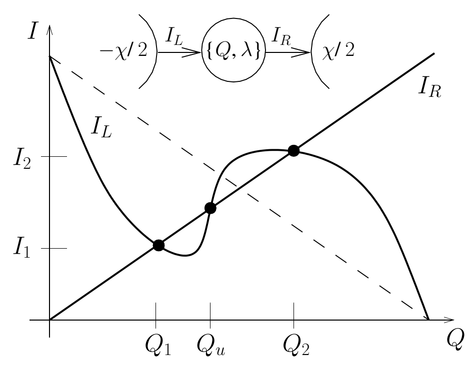

FIG. 1. A schematic of the average current flowing through the left and right connectors as a function of the charge in the central node. While linear connectors (the dashed line) have only one crossing point, nonlinear devices may have three crossing points, two stable, $Q_{1,2}$, and one unstable, Q_u. Inset: the stochastic network. The direction of current flow through the network is indicated.

states in the extreme small or large current ranges. In between, the distribution has a comparatively flat region dominated by the telegraph process, Eq. (15). Given $\Gamma_{1,2}$, it is universal: The logarithm of the distribution is a tilted ellipse.

The model.—The bistable network we consider consists of one "conserving" node, C, storing the generalized charge Q. A typical nonequilibrium fluctuation decays on a time scale τ_C. This node is connected to two "absorbing" nodes, L and R, which inject and absorb currents $\tilde{I}_L$ and $\tilde{I}_R$ through the connectors between the nodes (see inset of Fig. 1). The sources of noise are the currents $\tilde{I}_{L,R}$, which are Markovian random variables (after a time scale $\tau_0 \ll \tau_C$) with known statistics given by the current cumulant generating functions,

$$H_\alpha(\lambda_\alpha) = \sum_n (\lambda_\alpha^n/n!)\langle\langle \tilde{I}_\alpha^n \rangle\rangle, \qquad \alpha = L, R, \tag{1}$$

where the cumulants $\langle\langle \tilde{I}_\alpha^n \rangle\rangle$ are functions of the charge in the central node, Q [17]. Charge conservation, $\dot{Q} = \tilde{I}_L - \tilde{I}_R$, leads to the slow dynamics of Q which couples back into the dynamics of $\tilde{I}_{L,R}$ by changing the conditions for transport through the connectors, thus altering the statistics of the net current $I = (\tilde{I}_L + \tilde{I}_R)/2$. The cumulants $\langle\langle \mathcal{Q}^n \rangle\rangle$ of the *transmitted charge* $\mathcal{Q}(t) = \int_0^t dt' I(t')$ can be obtained with diagrammatic perturbation theory [12,16] in the stable regime by using the SPI formalism. However, the power of the SPI method is that it also allows for a nonperturbative analysis of unstable systems.

The SPI represents the time evolution of the system while monitoring all fluctuations [15,16]:

$$U[\chi, t] = \int \mathcal{D}Q\mathcal{D}\lambda \exp\left\{\int_0^t dt'[-\lambda\dot{Q} + H(Q, \lambda, \chi)]\right\}, \tag{2}$$

with fixed initial charge $Q(0)$ and the final condition $\lambda(t) = 0$. The generating function $S(\chi, t)$ for the cumulants of the transmitted charge $\langle\langle \mathcal{Q}^n \rangle\rangle = d_\chi^n S(\chi, t)|_{\chi=0}$ is given by $S = \ln U[\chi, t]$. The path integral (2) has a canonical form, with the "Hamiltonian" H given by

$$H(Q, \lambda, \chi) = H_L(\lambda + \chi/2) + H_R(-\lambda + \chi/2), \tag{3}$$

where $H_{L,R}$ are the current cumulant generators (1). The counting variable λ is a Lagrange multiplier conserving the charge on the node C. A large parameter, the number of elementary charges participating in the transport, allows the saddle-point evaluation of the SPI (2) leading to the canonical equations of motion

$$\dot{Q} = \partial H/\partial\lambda, \qquad \dot{\lambda} = -\partial H/\partial Q. \tag{4}$$

The solution of these equations should be substituted into Eq. (2) in order to obtain the generating function S.

The bistability.—We first assume for simplicity that the sources $\tilde{I}_\alpha$, $\alpha = L, R$ are Gaussian with average current $I_\alpha(Q) = \langle \tilde{I}_\alpha \rangle$ and noise power $F_\alpha(Q) = \langle\langle \tilde{I}_\alpha^2 \rangle\rangle$ and then generalize to arbitrary noise generators (1). Fixing $\chi = 0$, we find that the unconditional dynamics of the system (where the transmitted charge is unmonitored) is governed by the Hamiltonian

$$H_0 = (I_L - I_R)\lambda + (1/2)(F_L + F_R)\lambda^2. \tag{5}$$

Therefore, the stationary state $\dot{Q} = \dot{\lambda} = 0$ is [according to Eq. (4)] given by $\lambda = 0$ and $I_L(Q) = I_R(Q)$, the conservation of average current. In the Ohmic regime, the average current is a linear function of the charge in the node (see Fig. 1), so the currents intersect only at one point. However, if we allow a nonlinear I-V, topological distortions of the curves allow three crossing points, giving three stationary solutions. We now show that while the outer two intersections are stable to perturbations, the central intersection is unstable.

First, we note that because the Hamiltonian (5) does not explicitly depend on time, it is an integral of motion: $H_0(Q, \lambda) = E$, where the "energy" determines a trajectory in phase space. On a time scale longer than the relaxation time, $t \gg \tau_C$, the dissipative dynamics of the system projects a Q-distributed initial state onto the zero energy lines, $H_0(Q, \lambda) = 0$. According to Eq. (5), there are at least two such lines: the λ_0 line with $\lambda = 0$, and the "instanton" line $\lambda_{\rm in}$, given by

$$\lambda_{\rm in}(Q) = -2(I_L - I_R)/(F_L + F_R). \tag{6}$$

These lines, shown in Fig. 2, connect the three stationary points, Q_1, Q_u, and Q_2. From Eqs. (4), for the line λ_0 we have $\dot{Q} = I_L - I_R$, while for the $\lambda_{\rm in}$ line the sign is opposite, $\dot{Q} = I_R - I_L$. Then the simple investigation of Fig. 1 shows the propagation direction.

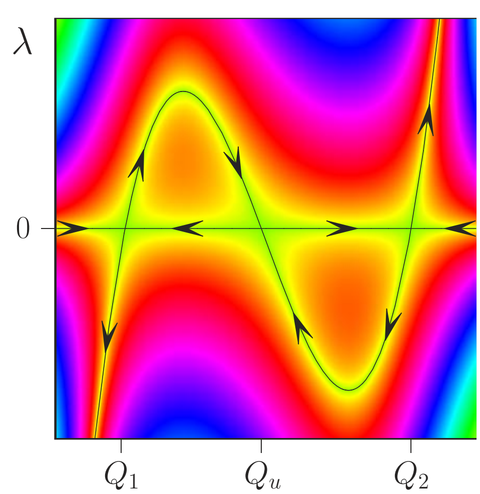

FIG. 2 (color online). The stability profile. The Hamiltonian is given as a phase space density plot. The system has three fixed points, two stable at $Q = Q_1, Q_2$ and one unstable at $Q = Q_u$. For long time, the system is forced to the zero energy lines in black, λ_0, $\lambda_{\rm in}$. A decay from the unstable state may happen with zero action, but to hop from either stable state to the unstable state requires finite action, the area underneath the curves shown.

According to Eq. (2), the action on the zero energy lines is given by $S = -\int \lambda dQ$. On the λ_0 line, the action vanishes, which makes the dynamics along this line most probable. This dynamics describes the relaxation of the initial state to the points Q_1 and Q_2 (see Fig. 2), making them the stable stationary points of the system, and Q_u the unstable point. The propagation along the $\lambda_{\rm in}$ line away from stable points generates a nonzero action, and therefore it has an exponentially small probability. Occasionally, the system switches between points Q_1 and Q_2 on the lines λ_0, $\lambda_{\rm in}$ with the action $S = -A_{1,2}$,

$$A_{1,2} = \int_{Q_{1,2}}^{Q_u} dQ \lambda_{\rm in}(Q), \tag{7}$$

given in Fig. 2 by the area between the lines λ_0 and $\lambda_{\rm in}$.

For an estimate, we consider the simplest (symmetric cubic) current difference, $I_L - I_R = -\alpha Q[Q^2 - (\Delta Q)^2]$, and constant noise, $F_L + F_R = 2\bar{F}$. For the action (7) with the instanton (6), we obtain $A_{1,2} = (\alpha/4\bar{F})(\Delta Q)^4$. By setting $A_{1,2} \sim 1$ and using Eqs. (4) we estimate the switching time $\tau_{\rm in} \sim (\Delta Q)^2/\bar{F}$, a new time scale in the problem. The assumption of effectively continuous charge demands that $\Delta Q \gg 1$, leading to $\tau_{\rm in} \gg \tau_0$, so that the separation of time scales requirement is satisfied. This also implies the instability is well developed.

The result (7) derived above for Gaussian noise with the instanton (6) holds for general noise (1) with the $\lambda_{\rm in}$ line defined by the equation

$$H_0(Q, \lambda) = H_L(\lambda) + H_R(-\lambda) = 0. \tag{8}$$

From now on we consider general noise $H_{L,R}$ [18].

The effective two-state Hamiltonian.—We now proceed with the instanton calculation [19]. We consider a time interval Δt much longer than the switching time $\tau_{\rm in}$. On this time scale, the system always relaxes to one of the stable points $Q_{1,2}$, so that it can be considered as being effectively a two-state system where the evolution operator (2) becomes a 2×2 matrix. Starting from one of the stable points, the system will switch to the other stable point with the small transition probability $\Gamma_{1,2}\Delta t$ that is proportional to the time interval, because the attempts are uncorrelated. The switching rates are given by the instanton ($\lambda_{\rm in}$-line) contribution to the path integral,

$$\Gamma_{1,2} = \omega_{1,2} \exp(-A_{1,2}), \tag{9}$$

where the action $A_{1,2}$ is given by Eq. (7) [20]. We note if the noise goes to zero, then $\Gamma_{1,2}$ vanish exponentially. The prefactor $\omega_{1,2}$ is the attempt rate, which is subdominant [22]. Finally, using the smallness of the switching rates, we can write a differential master equation for the evolution operator, $\dot{U} = \hat{H}_0 U$, where the effective Hamiltonian matrix has components $\pm\Gamma_{1,2}$. The evolution operator describes the relaxation with the rate $\Gamma_S = \Gamma_1 + \Gamma_2$ of any given initial state to the stationary state with constant occupation probabilities,

$$P_1 = \Gamma_2/\Gamma_S, \qquad P_2 = \Gamma_1/\Gamma_S. \tag{10}$$

Having described the bare (unconditional) dynamics of the system, we now concentrate on the transport statistics. A χ contribution to the Hamiltonian (5) deforms the stability profile in Fig. 2. The stable stationary points acquire a nonzero energy and thus contribute to the diagonal values in the full effective Hamiltonian $\hat{H}$. The rates $\Gamma_{1,2}$ will also depend on χ. However, the bistability is important only for $\chi \propto \Gamma_{1,2}$, and therefore the χ correction to $\Gamma_{1,2}$ is exponentially small and may be neglected. Using Eq. (3) to evaluate the contribution of the stable points to the action, we arrive at the result

$$\hat{H} = \begin{pmatrix} H_1(\chi) - \Gamma_1 & \Gamma_2 \\ \Gamma_1 & H_2(\chi) - \Gamma_2 \end{pmatrix}. \tag{11}$$

The Hamiltonians $H_{1,2}(\chi)$ are the noise generators at the stable points $Q_{1,2}$ with the net currents $I_{1,2} = I_L = I_R$ and noise power $F_{1,2} = (F_L G_R^2 + F_R G_L^2)/(G_L + G_R)^2$, where $G_{L,R}$ are the differential conductances taken at $Q_{1,2}$. High cumulants may be found perturbatively [16].

The transport statistics.—The Hamiltonian (11) determines the conditional dynamics of the effective two-state system on the time scale $t \gg \tau_{\rm in}$ [23]. It describes the relaxation of the system to the new *conditional* stationary state. In an experiment on the bistable system, if the averaging time of the measurement is much shorter than the relaxation time $\tau_S = (\Gamma_1 + \Gamma_2)^{-1}$, the output of the measurement device will reveal the noise properties of only one of the states. On the time scale $t \gg \tau_S$ the system switches many times and loses its correlation with the initial state. As a result, the transport becomes again Markovian, $S(\chi) = tH(\chi)$. The generating function $H(\chi)$ is given by the largest eigenvalue of the Hamiltonian (11),

$$H(\chi) = (1/2) \sum_{n=1,2} (H_n - \Gamma_n) + \sqrt{(H_2 - H_1 - \Delta\Gamma)^2/4 + \Gamma_1\Gamma_2}, \tag{12}$$

where $\Delta\Gamma = \Gamma_2 - \Gamma_1$. We immediately see that the distribution is properly normalized because $H(0) = 0$. The average current is given by $\langle I \rangle = H'(0) = \sum_{n=1,2} I_n P_n$, and the noise power $\langle\langle I^2 \rangle\rangle = H''(0)$ by

$$\langle\langle I^2 \rangle\rangle = \sum_{n=1,2} F_n P_n + 2(\Delta I)^2 \Gamma_1 \Gamma_2/\Gamma_S^3, \tag{13}$$

where $\Delta I = I_2 - I_1$, and the stationary probabilities P_n are given in Eq. (10). The first term is the weighted noise of the stationary states, and the second term is the well-known result for zero-frequency telegraph noise [5] which dominates the first term because the rates are small. The dominant contribution to the third cumulant is $\langle\langle I^3 \rangle\rangle = 6(\Delta I)^3 \Gamma_1 \Gamma_2 \Delta\Gamma/\Gamma_S^5$. The slow switching rates cause all cumulants to be parametrically large, making bistable

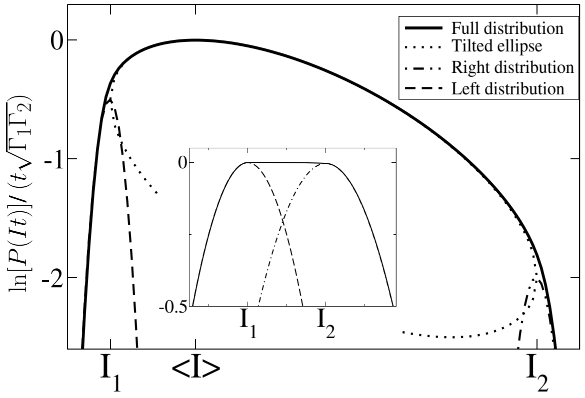

FIG. 3. The log distribution of transmitted charge, divided by $t\sqrt{\Gamma_1\Gamma_2}$. Within the bistable range, the distribution is well fit by a tilted ellipse, connecting to the distribution dominated by state 1 or 2. The average current is denoted by $\langle I\rangle$. Inset: the same distribution, $\ln P[It]/t$, showing the comparatively flat bistable distribution with the left and right asymptotics. $\Gamma_2/\Gamma_1 = 4$, $F_{1,2} = (1, 1.5) \times 10^{-2}\langle\langle I^2\rangle\rangle$.

systems a promising candidate for an experimental measurement of the transport statistics.

We now Fourier transform $U(i\chi)$ using the stationary phase approximation to obtain the probability to transmit charge $\mathcal{Q} = It$ through the system in time t. Outside the bistability region, $I < I_1$ or $I > I_2$, the generator in Eq. (12) may be replaced with $H_1 - \Gamma_1$ or $H_2 - \Gamma_2$, so that the extreme value statistics is dominated by one of the stable states. Within the bistable region, $I_1 < I < I_2$, the probability $\mathcal{P}_n(I)$ of occupying state $n = 1, 2$ under the condition that a charge It is transmitted is given by

$$\mathcal{P}_{1,2}(I) = \mathcal{G}_{2,1}/(\mathcal{G}_1 + \mathcal{G}_2), \quad \mathcal{G}_n = \sqrt{\Gamma_n|I - I_n|}, \quad (14)$$

which generalizes (10). The probability distribution of transmitted charge is given by

$$\ln P(It)/t = -(\mathcal{G}_1 - \mathcal{G}_2)^2/\Delta I \quad (15)$$

and is determined by the telegraph process. The logarithm of the exact distribution (divided by the rates) is shown in Fig. 3 inside the bistable region and is well fit by a tilted ellipse, while the inset compares left and right asymptotics to the full distribution.

In conclusion, we have discussed a general model of a noise driven bistable system. We have calculated the switching rates (9), transport statistics (12) and (15), and conditional occupation probabilities (14). The higher cumulants in Eq. (1) determine the instanton lines in Fig. 2 [see Eq. (8)] and thus make an exponentially large contribution to the rates $\Gamma_{1,2}$ [7,18]. Away from the bistable region, the probability tails decay with a non-Gaussian distribution determined by the isolated statistics $H_{1,2}$ of the stable states. Within the bistable current range, the distribution is always an ellipse whose dimensions are fixed by the rates alone, showing a robust universality.

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