Remote Preparation and Distribution of Bipartite Entangled States

Gilad Gour[1,2,*] and Barry C. Sanders[3]

[1]Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1, Canada
[2]Department of Mathematics, University of California–San Diego, La Jolla, California 92093-0112, USA
[3]Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada

(Received 16 July 2004; published 20 December 2004)

We prove a powerful theorem for tripartite remote entanglement distribution protocols that establishes an upper bound on the amount of entanglement of formation that can be created between two single-qubit nodes of a quantum network. Our theorem also provides an operational interpretation of concurrence as a type of entanglement capacity.

DOI: 10.1103/PhysRevLett.93.260501 PACS numbers: 03.67.Hk, 03.65.Ud, 03.67.Mn

Shared bipartite entanglement is a crucial shared resource for many quantum information tasks such as teleportation [1], entanglement swapping [1–4], and remote state preparation (RSP) [5] that are employed in quantum information protocols. In general, different parties, or nodes, of a quantum network (QNet) share an entanglement resource, such as ebits (maximally entangled pure bipartite states), which are consumed during the task. In practice, generating entangled states is expensive, but here we establish a protocol by which a QNet requires only a single supplier of entanglement to all nodes who, by judicious measurements and classical communication, provides the nodes with a unique pairwise entangled state independent of the measurement outcome. Furthermore, we extend this result to a chain of suppliers and nodes, which enables an operational interpretation of concurrence [6].

In the special case that the supplier (whom we call "Sapna") shares bipartite states with two nodes (labeled "Alice" and "Bob"), and such states are pure and maximally entangled, our protocol corresponds to entanglement swapping. However, in the practical case that initial shared entanglement between suppliers and nodes involves partially entangled or mixed states, we show that general local operations and classical communication (LOCC) by *all* parties (suppliers and nodes) yields *distributions* of entangled states between nodes. In general, a distribution of bipartite entangled states between any two nodes includes states that do not have the *same* entanglement (i.e., not all the states are equivalent under LOCC between the nodes); thus we name this general process remote entanglement distribution (RED). In our terminology entanglement swapping with partially entangled states [4] is a particular class of RED protocols. Here we identify which distributions of states (shared between Alice and Bob) can or cannot be created by RED. In particular, we prove a powerful theorem that establishes, for the (2×2)-dimensional mixed case, an upper bound on the entanglement of formation that can be produced between Alice and Bob. We extend this result to the case of a linear chain of parties that plays the role of suppliers and nodes; this extension provides an operational interpretation of concurrence.

Then we discuss an especially interesting class of tripartite RED protocols in which Alice and Bob (after LOCC by the three parties) end up sharing a *unique* bipartite entangled state, rather than a distribution of entangled states. In this scheme, Sapna not only wishes to create entanglement between Alice and Bob, but she also wishes to provide Alice and Bob with a single entangled state (which, in general, is unknown to Alice and Bob [7]). When the initial bipartite states (shared between Sapna and the two nodes) are partially entangled $d \times d$ pure states, or belong to a particular nontrivial class of mixed states, we provide a protocol for Sapna to remotely prepare a bipartite entangled state between Alice and Bob. In this protocol, Sapna performs a single orthogonal (von Neumann) measurement, then transmits $\log_2 d$ bits of classical information to Alice and $2\log_2 d$ bits to Bob. Based solely on the classical information received from Sapna, Alice and Bob perform local unitary operations to obtain the state that Sapna intends them to share. Our protocol for remote preparation of bipartite entangled states (RPBES) works even when entanglement is insufficient for Sapna to simply teleport qubits to Alice and Bob.

Our scheme for tripartite RED (including tripartite RPBES) commences with a four-way shared state, $\hat{\rho}_{1234} = \hat{\rho}_{12} \otimes \hat{\rho}_{34}$ for $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ bipartite entangled states, and with Sapna holding shares 2 and 3, and Alice and Bob holding shares 1 and 4, respectively. Each share has a d-dimensional Hilbert space. Alice, Bob, and Sapna perform general LOCC (allowing classical communication among *all* three parties) to create outcomes $\mathcal{O} \equiv \{\hat{\sigma}^j_{14} = \mathrm{Tr}_{23}\hat{\sigma}^j_{1234}, Q_j; j = 1, \ldots, s\}$ with Q_j the probability that Alice and Bob share the state $\hat{\sigma}^j_{14}$, which is obtained by reducing the four-way state $\hat{\sigma}^j_{1234}$ over Sapna's shares.

In the case of RPBES, $\hat{\sigma}^j_{14}$ represents the state obtained after a *single* measurement performed by Sapna. Then, after Sapna broadcasts the measurement result j, Alice and Bob each perform a single local unitary operation to transform $\hat{\sigma}^j_{14}$ into a unique entangled state (i.e., independent of j). This scheme for RPBES is always possible if $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ are partially entangled pure states or belong to a particular nontrivial class of mixed states as we show.

Let us begin by proving an important theorem that rules out certain distributions (of bipartite states) from being able to be created by general tripartite RED: this restriction is obtained via a bound for the average concurrence of the resultant distribution shared by Alice and Bob in relation to the concurrences of the initial states Sapna has shared with both Alice and Bob. Concurrence for a pure bipartite state $|\psi\rangle$ is $C(|\psi\rangle) \equiv \sqrt{2(1 - \mathrm{Tr}\hat{\rho}_r^2)}$ [6,8,9] (with ρ_r obtained by tracing the pure-state density matrix $|\psi\rangle\langle\psi|$ over one of the two shares). Concurrence for a mixed state $\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is defined as the average concurrence of the pure states of the decomposition, minimized over all decompositions of $\hat{\rho}$ (the convex roof): $C(\hat{\rho}) = \min \sum_i p_i C(|\psi_i\rangle)$. (The concurrence for an arbitrary two-qubit state has been calculated [6], and a concurrence lower bound for higher dimensions has been established [9].)

Theorem 1. If Alice, Bob, and Sapna perform general LOCC on the initial four-qubit state $\hat{\rho}_{12} \otimes \hat{\rho}_{34}$ with outcome $\{Q_j, \hat{\sigma}^j_{14}\}$, then

$$C_{14} \equiv \sum_{j=1}^{s} Q_j C(\hat{\sigma}^j_{14}) \le C_{12} C_{34}, \tag{1}$$

with $C_{12} \equiv C(\hat{\rho}_{12})$ and $C_{34} \equiv C(\hat{\rho}_{34})$.

Proof. In the optimal decompositions $\hat{\rho}_{12} = \sum_{l=1}^4 p_l |\psi^{(l)}\rangle_{12}\langle\psi^{(l)}|$, $\hat{\rho}_{34} = \sum_{l=1}^4 q_l |\chi^{(l)}\rangle_{34}\langle\chi^{(l)}|$, we can always choose optimal decompositions such that the four states $|\psi^{(l)}\rangle_{12}$ have the same concurrence C_{12} and all four states $|\chi^{(l)}\rangle_{34}$ have the same concurrence C_{34} [6]. Thus, Schmidt coefficients of the states $|\psi^{(l)}\rangle_{12}$ and $|\chi^{(l)}\rangle_{34}$ do not depend on the index l:

$$\begin{aligned} |\psi^{(l)}\rangle_{12} &= \sqrt{\lambda_0}|0^{(l)}0^{(l)}\rangle_{12} + \sqrt{\lambda_1}|1^{(l)}1^{(l)}\rangle_{12}, \\ |\chi^{(l)}\rangle_{34} &= \sqrt{\eta_0}|0^{(l)}0^{(l)}\rangle_{34} + \sqrt{\eta_1}|1^{(l)}1^{(l)}\rangle_{34} \end{aligned} \tag{2}$$

with λ_i, η_i the Schmidt coefficients of $|\psi^{(l)}\rangle_{12}$, $|\chi^{(l)}\rangle_{34}$, respectively. The index l in the states $\{|0^{(l)}\rangle_i, |1^{(l)}\rangle_i\}$ represents four different bases for each system $i = 1, 2, 3, 4$. Note that in this notation $C_{12} = 2\sqrt{\lambda_0\lambda_1}$ and $C_{34} = 2\sqrt{\eta_0\eta_1}$.

Since the entanglement between Alice and Bob remains zero unless Sapna perform a measurement, we assume that the first measurement is performed by Sapna and is described by the Kraus operators $\hat{M}^{(j)}$ and their components $M^{(j,ll')}_{mm',kk'} \equiv {}_{23}\langle m^{(l)} m'^{(l')}|\hat{M}^{(j)}|k^{(l)}k'^{(l')}\rangle_{23}$, with $k, k', m, m' = 0, 1$ and $l, l' = 1, 2, 3, 4$. The density matrix shared between Alice and Bob after outcome j occurs is

$$\begin{aligned} \hat{\sigma}^{\,j}_{14} &= \frac{1}{Q_j}\mathrm{Tr}_{23}(\hat{M}^{(j)\dagger}\hat{\rho}_{12} \otimes \hat{\rho}_{23}\hat{M}^{(j)}) \\ &= \frac{1}{Q_j}\sum_{l,l'}\sum_{m,m'} p_l q_{l'} r^{(j,ll')}_{mm'} |\phi^{(i,ll')}_{mm'}\rangle_{14}\langle\phi^{(i,ll')}_{mm'}|, \end{aligned} \tag{3}$$

where $r^{(j,ll')}_{mm'} \equiv \sum_{k,k'} \lambda_k \eta_{k'} |M^{(j,ll')}_{mm',kk'}|^2$,

$$|\phi^{(j,ll')}_{mm'}\rangle_{14} \equiv \frac{1}{\sqrt{r^{(j,ll')}_{mm'}}} \sum_{k,k'} \sqrt{\lambda_k \eta_{k'}}\, M^{(j,ll')}_{mm',kk'} |k^{(l)} k'^{(l')}\rangle_{14}, \tag{4}$$

and $Q_j = \sum_{l,l',m,m'} p_l q_{l'} r^{(j,ll')}_{mm'}$ is the probability to obtain an outcome j. Now, a direct calculation of $C(|\phi^{(j,ll')}_{mm'}\rangle_{14})$ gives

$$\begin{aligned} C(|\phi^{(i,ll')}_{mm'}\rangle_{14}) = \frac{2\sqrt{\lambda_0\lambda_1\eta_0\eta_1}}{r^{(j,ll')}_{mm'}} |&M^{(j,ll')}_{mm',00} M^{(j,ll')}_{mm',11} \\ &- M^{(j,ll')}_{mm',01} M^{(j,ll')}_{mm',10}|. \end{aligned} \tag{5}$$

As the concurrence of $\hat{\sigma}^j_{14}$ cannot exceed the average concurrence of the decomposition in Eq. (3), we have

$$\begin{aligned} C_{14} &\equiv \sum_{j=1}^{s} Q_j C(\hat{\sigma}^j_{14}) \\ &\le 2\sqrt{\lambda_0\lambda_1\eta_0\eta_1}\sum_{l,l'} p_l q_{l'} \sum_j \sum_{m,m'} |M^{(j,ll')}_{mm',00} M^{(j,ll')}_{mm',11} \\ &\quad - M^{(j,ll')}_{mm',01} M^{(j,ll')}_{mm',10}| \\ &\le \frac{1}{4} C_{12} C_{34} \sum_j \mathrm{Tr}(\hat{M}^{(j)\dagger}\hat{M}^{(j)}), \end{aligned} \tag{6}$$

where the last inequality follows from the fact that $|ab - cd| \le (|a|^2 + |b|^2 + |c|^2 + |d|^2)/2 \ \forall\ a, b, c, d \in \mathbb{C}$. The completeness relation, $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = I$, implies Eq. (1).

Consider now the following LOCC: after Sapna's first measurement, she sends the result j to Alice and Bob. Based on this result, Alice then performs a measurement represented by the Kraus operators $\hat{A}^{(k)}_j$ and sends the result k to Bob and Sapna. Based on the results j, k from Sapna and Alice, Bob performs a measurement represented by the Kraus operators $\hat{B}^{(n)}_{jk}$ and sends the result n to Sapna. In the last step of this scheme, Sapna performs a second measurement with Kraus operators denoted by $\hat{F}^{(j)}_{jkn}$ and sends the result i to Alice and Bob. The final distribution of states shared by Alice and Bob is $\{N_{jkni}, \hat{\sigma}^{jkni}_{14}\}$, where N_{jkni} is the probability for outcome j, k, n, i.

As the concurrence of arbitrary $|\psi\rangle_{14}$ satisfies $C(\hat{A}^{(k)}_j \otimes \hat{B}^{(n)}_{jk}|\psi\rangle) = |\mathrm{Det}(\hat{A}^{(k)}_j)||\mathrm{Det}(\hat{B}^{(n)}_{jk})|C(|\psi\rangle)$, Eq. (6) yields

$$\begin{aligned} C_{14} &\equiv \sum_{j,k,n,i} N_{jkni} C(\hat{\sigma}^{jkni}_{14}) \\ &\le \frac{1}{4} C_{12} C_{34} \sum_{j,k} |\mathrm{Det}(\hat{A}^{(k)}_j)| \sum_n |\mathrm{Det}(\hat{B}^{(n)}_{jk}) \\ &\quad \times |\sum_i \mathrm{Tr}(\hat{M}^{(j)\dagger}\hat{F}^{(i)\dagger}_{jkn}\hat{F}^{(i)}_{jkn}\hat{M}^{(j)}). \end{aligned} \tag{7}$$

Moreover, from the geometric-arithmetic inequality we have $\sum_n |\mathrm{Det}(\hat{B}^{(n)}_{jk})| \le \frac{1}{2}\sum_n \mathrm{Tr}(\hat{B}^{(n)\dagger}_{jk}\hat{B}^{(n)}_{jk}) = 1$ and a similar relation for $\hat{A}_j^{(k)}$. These results, together with $\sum_i \hat{F}_{jkn}^{(i)\dagger}\hat{F}_{jkn}^{(i)} = 1$, lead us back to Eq. (6). Evidently, all operations by Alice, Bob, and Sapna after Sapna's first measurement cannot increase the bound on C_{14}. □

Theorem 1 concerns one supplier and two nodes, but, in fact, applies to one supplier and *any* pair of nodes; thus, the result of Theorem 1 is applicable to an arbitrarily large QNet with one supplier and many nodes. In fact, Theorem 1 can be extended to more than one supplier, as stated in the following corollary.

Corollary 1. Consider an aligned chain of N mixed bipartite two-qubit states, $\hat{\rho}_{01}, \hat{\rho}_{12}, \ldots, \hat{\rho}_{N-1N}$, where the state $\hat{\rho}_{k-1k}$ $(k = 1, 2, \ldots, N)$ is shared between party $k-1$ and party k. If the $N+1$ parties perform LOCC on the initial state $\hat{\rho}_{01} \otimes \hat{\rho}_{12} \otimes \cdots \otimes \hat{\rho}_{N-1N}$ with the resultant distribution of states between party 0 and N denoted by $\{P_j, \hat{\sigma}_{0N}^j\}$ (P_j is the probability to have the state $\hat{\sigma}_{0N}^j$), then

$$C_{0N} \equiv \sum_j P_j C_d(\hat{\sigma}_{0N}^j) \leq C_{01}C_{12}\cdots C_{N-1N}, \qquad (8)$$

with $C_{k-1k} \equiv C(\hat{\rho}_{k-1k})$ $(k = 1, 2, \ldots, N)$.

Theorem 1 and its corollary suggest an interpretation of the concurrence as a form of *entanglement capacity.* Until now concurrence has served as a powerful mathematical tool, but here we have introduced an operational description of the concurrence. Furthermore, for two qubits, concurrence is equivalent to entanglement of formation: our theorem establishes an upper bound to the average entanglement of formation that the supplier can create.

In the following, we show that the equality in Eq. (1) can always be achieved if both $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ are partially entangled and pure. Saturation of the bound is also possible if one of the states is maximally entangled and the other is any mixed state (in which case the bound is saturated via quantum teleportation [1]). Later we provide an example showing that the bound saturates in some cases for one state mixed and the other a *partially* entangled pure state. It is not known, however, if saturation is always achievable.

Proposition 1. The equality in Eq. (1) can always be achieved by RPBES if $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ are both 2×2 bipartite pure states. In order to prove that the equality in Eq. (1) is achievable for pure states, we establish a protocol for RPBES taking first $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ to be $d \times d$ pure states. For $d = 2$ our protocol saturates inequality (1).

Working with partially entangled states is important because in the nonasymptotic regime the process of concentration is expensive, and it is less expensive in terms of ebits consumed (as well as classical bits [10]) to work directly with partially entangled states [11]). The protocol below enables Sapna to control the amount of entanglement shared between Alice and Bob. In the (2×2)-dimensional (pure) case the concurrence uniquely determines the entanglement of the bipartite state. In this case maximum concurrence corresponds to maximum possible entanglement. However, for $d > 2$ (or for mixed states), the concurrence of a $(d \times d)$-bipartite (partially) entangled state is not sufficient to determine all the Schmidt coefficients. Thus, in this case, the optimal bipartite state that can be prepared by Sapna is not unique. It depends on the choice taken for the measure of entanglement; therefore, Sapna remotely prepares entangled states according to the tasks Alice and Bob need to perform.

The RPBES protocol.—Let the two pure densities be expressed as $\hat{\rho}_{12} = |\psi\rangle_{12}\langle\psi|$ and $\hat{\rho}_{34} = |\chi\rangle_{34}\langle\chi|$ with $|\psi\rangle_{12}$ and $|\chi\rangle_{34}$ states in d^2-dimensional Hilbert spaces. The initial states $|\psi\rangle_{12}$ and $|\chi\rangle_{34}$ are expressed in the Schmidt decomposition as $|\psi\rangle_{12} = \sum_{k=0}^{d-1}\sqrt{\lambda_k}|kk\rangle_{12}$ and $|\chi\rangle_{34} = \sum_{k=0}^{d-1}\sqrt{\eta_k}|kk\rangle_{34}$. The steps of the protocol are as follows. (i) Sapna performs a projective measurement $\hat{P}^{(j,j')} = |P^{(j,j')}\rangle_{23}\langle P^{(j,j')}|$, $j, j' = 0, 1, \ldots, d-1$,

$$|P^{(j,j')}\rangle_{23} \equiv \frac{1}{d}\sum_{m,m'=0}^{d-1} e^{i[(2\pi/d^2)(dj+j')(dm+m')+\theta_{mm'}]}|mm'\rangle_{23}, \qquad (9)$$

with $\theta_{mm'} \in \mathbb{R}$ chosen freely. Note that the d^2 states $|P^{(j,j')}\rangle_{23}$ are orthonormal, regardless of the choice of $\theta_{mm'}$. (ii) After the outcomes j, j' have been obtained, the system state becomes $|P^{(j,j')}\rangle_{23}|\phi^{(j,j')}\rangle_{14}$, where

$$|\phi^{(j,j')}\rangle_{14} = \sum_{m=0}^{d-1}\sum_{m'=0}^{d-1}\sqrt{\lambda_m\eta_{m'}} \times e^{-i[(2\pi/d^2)(dj+j')(dm+m')+\theta_{mm'}]} \times |mm'\rangle_{14}. \qquad (10)$$

(iii) Sapna sends the results j and j' to Bob ($2\log_2 d$ bits of information) and the result j' ($\log_2 d$ bits of information) to Alice. Bob then performs the unitary operation

$$\hat{U}_b^{(j,j')}|m'\rangle_4 = \exp\left[i\frac{2\pi}{d^2}(dj+j')m'\right]|m'\rangle_4, \qquad (11)$$

and Alice performs the unitary operation

$$\hat{U}_a^{(j')}|m\rangle_1 = \exp\left(i\frac{2\pi}{d}j'm\right)|m\rangle_1. \qquad (12)$$

(iv) The final state shared between Alice and Bob is [12]

$$|F\rangle_{14} = \sum_{m=0}^{d-1}\sum_{m'=0}^{d-1}\exp(-i\theta_{mm'})\sqrt{\lambda_m\eta_{m'}}|mm'\rangle_{14}. \qquad (13)$$

For $d = 2$, Proposition 1 is proved by taking $\theta_{mm'} = \pi mm'$; in this case the concurrence of $|F\rangle_{14}$ equals $4\sqrt{\lambda_0\lambda_1\eta_0\eta_1} = C_{12}C_{34}$, which is optimal (see Theorem 1). Moreover, if Alice and Bob know the state prepared by Sapna, they can perform local unitaries to obtain any state with the same concurrence. Thus, *any* (2×2)-dimensional bipartite pure state with concurrence not greater than $C_{12}C_{34}$ can be prepared by Alice, Bob, and Sapna performing LOCC.

For $d > 2$, if all λ_m and $\eta_{m'}$ in Eq. (13) are equal to $1/d$, then the choice $\theta_{mm'} = 2\pi mm'/d$ gives a maximally entangled state. For different λ_m and $\eta_{m'}$, the optimal bipartite state that can be prepared by Sapna depends on the particular entanglement measure. For example, the concurrence of the $(d \times d)$ bipartite state in Eq. (13) is

$$C(|F\rangle_{14}) = 2\Bigg\{\sum_{k>k'}\sum_{m>m'} \lambda_k \lambda_{k'} \eta_m \eta_{m'} |e^{i(\theta_{km}+\theta_{k'm'})} - e^{i(\theta_{km'}+\theta_{k'm})}|^2\Bigg\}^{1/2}. \tag{14}$$

Unlike the 2×2 case, for $d > 2$ the term with the absolute value in Eq. (14) cannot equal 2 for all k, k', m, m'. Thus, the values of θ_{km} that maximize $C(|F\rangle_{14})$ depend explicitly on the Schmidt coefficients λ_k and η_m.

Our protocol can be applied for mixed states. In general, for mixed states $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$, our protocol provides Alice and Bob with a *distribution* of mixed states rather then a unique state. As (for RPBES) Sapna wishes to produce a unique state $\hat{\sigma}_{14}$, we establish a class of mixed bipartite states $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$ where our protocol yields a unique state. We then provide a specific (2×2)-dimensional example, which we show is optimal (maximum possible concurrence for the state shared by Alice and Bob).

The two initial $(d \times d)$ bipartite density matrices are $\hat{\rho}_{12} = \sum_{l=1}^{n} p_l |\psi^{(l)}\rangle_{12}\langle\psi^{(l)}|$, $\hat{\rho}_{34} = \sum_{l'=1}^{n'} q_{l'} |\chi^{(l')}\rangle_{34}\langle\chi^{(l')}|$, with $n, n' \leq d$, and $|\psi^{(l)}\rangle_{12} = \sum_{k=0}^{d-1} a_k^{(l)} |kk\rangle_{12}$, $|\chi^{(l')}\rangle_{34} = \sum_{k'=0}^{d-1} b_{k'}^{(l')} |k'k'\rangle_{34}$, with $a_k, b_{k'} \in \mathbb{C}$ and basis states $|k\rangle_i$ independent of l and l'; this characterizes the class containing $\hat{\rho}_{12}$ and $\hat{\rho}_{34}$.

Now, it can be shown that, after Sapna performs her measurement, and Bob and Alice perform the unitary operations of Eqs. (11) and (12), the resultant shared state is

$$\hat{\sigma}_{14} = \sum_{l=1}^{n}\sum_{l'=1}^{n'} p_l q_{l'} |\phi^{(ll')}\rangle_{14}\langle\phi^{(ll')}|, \tag{15}$$

with $|\phi^{(ll')}\rangle_{14} = \sum_{k=0}^{d-1}\sum_{k'=0}^{d-1} a_k^{(l)} b_{k'}^{(l')} e^{-i\theta_{kk'}} |kk'\rangle_{14}$.

We conclude with a simple interesting example. Suppose Alice shares with Bob the (2×2)-dimensional pure state $|\psi\rangle_{12} = \sqrt{\lambda_0}|00\rangle_{12} + \sqrt{\lambda_1}|11\rangle_{12}$ and Sapna shares with Bob the (2×2)-dimensional mixed state $\hat{\rho}_{34} = q|\chi^{(+)}\rangle_{34}\langle\chi^{(+)}| + (1-q)|\chi^{(-)}\rangle_{34}\langle\chi^{(-)}|$, where $0 \leq q \leq 1$ and $|\chi^{(\pm)}\rangle_{34} = (1/\sqrt{2})(|00\rangle_{34} \pm |11\rangle_{34})$. The concurrence of $|\psi\rangle_{12}$ is $2\sqrt{\lambda_0\lambda_1}$ and the concurrence of $\hat{\rho}_{34}$ is equal to $|2q-1|$ [6]. It is easy to see that both $|\psi\rangle_{12}$ and $\hat{\rho}_{34}$ belong to the class of density matrices described above. In this example, it is possible to calculate the concurrence of the final mixed state $\hat{\sigma}_{14}$ given in Eq. (15): $C_{14} = |2q - 1|\sqrt{\lambda_0\lambda_1}|e^{i(\theta_{00}+\theta_{11})} - e^{i(\theta_{01}+\theta_{10})}|$. For $\theta_{kk'} = \pi kk'$, $k, k' = 0, 1$, we obtain $C_{14} = C_{12}C_{34}$: the bound in Theorem 1 is saturated. The protocol is optimal in this example, and Bob can prepare the bipartite state $\hat{\sigma}_{14}$ with *any* concurrence between 0 and $C_{12}C_{34}$.

In summary, we have introduced a protocol for a QNet that allows a single supplier, who first shares entanglement with all nodes of the QNet (which may be partially entangled pure states or a particular class of mixed states), to provide any pair of nodes in the QNet with a single bipartite entangled state. We have also proved a powerful theorem for tripartite RED protocols that establishes an upper bound on the amount of entanglement of formation that can be created between two single-qubit nodes of the QNet. We have also proven that it is possible (in some cases) to saturate the concurrence bound in the theorem if one state is pure (even if it is partially entangled), and the other is mixed. Our theorem provides an operational interpretation of concurrence as an entanglement capacity.

G. G. acknowledges support by the Killam Trust, the DARPA QuIST program under Contract No. F49620-02-C-0010, and the NSF under Grant No. ECS-0202087. B. C. S. acknowledges support from iCORE and CIAR.

---

*Electronic address: ggour@math.ucsd.edu

[1] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[2] M. Żukowski *et al.*, Phys. Rev. Lett. **71**, 4287 (1993).
[3] S. Bose *et al.*, Phys. Rev. A **57**, 822 (1998); **60**, 194 (1999); B.-S. Shi *et al.*, Phys. Rev. A **62**, 054301 (2000).
[4] L. Hardy and D. D. Song, Phys. Rev. A **62**, 052315 (2000).
[5] C. H. Bennett *et al.*, Phys. Rev. Lett. **87**, 077902 (2001); M.-Y. Ye *et al.*, Phys. Rev. A **69**, 022310 (2004); D. W. Berry, Phys. Rev. A **70**, 062306 (2004).
[6] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[7] However, if they do know the state, Sapna, Alice, and Bob will be able to prepare a larger class of states.
[8] P. Rungta *et al.*, Phys. Rev. A **64**, 042315 (2001).
[9] F. Mintert *et al.*, Phys. Rev. Lett. **92**, 167902 (2004).
[10] G. Gour, Phys. Rev. A **70**, 042301 (2004).
[11] RSP with partially entangled states has been studied [5], but here we are concerned with four-node sharing: Sapna with two shares and Alice and Bob each with one share.
[12] The state $|F\rangle_{14}$ is separable for $\theta_{mm'} = 0$.