Entrainment of Randomly Coupled Oscillator Networks by a Pacemaker

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Entrainment by a pacemaker, representing an element with a higher frequency, is numerically investigated for several classes of random networks which consist of identical phase oscillators. We find that the entrainment frequency window of a network decreases exponentially with its depth, defined as the mean forward distance of the elements from the pacemaker. Effectively, only shallow networks can thus exhibit frequency locking to the pacemaker. The exponential dependence is also derived analytically as an approximation for large random asymmetric networks.

PACS numbers: 05.45.Xt, 87.18.Sn, 89.75.Fb

Pacemakers are wave sources in distributed oscillatory systems typically associated with a local group of elements having a higher oscillation frequency. Target patterns, generated by pacemakers, were the first complex wave patterns observed in the Belousov-Zhabotinsky system [1]. Pacemakers play an important role in the functioning of the heart [2] and in the collective behavior of *Dictyostelium discoideum* [3]. They are also observed in large-scale ecosystems [4]. In addition to pacemakers produced by local heterogeneities in the medium [5], self-organized pacemakers in uniform birhythmic media have been theoretically studied [6]. While the majority of related investigations have so far been performed for systems with local diffusive coupling between the elements, pacemakers can also operate in oscillator networks with complex connection topologies. The circadian rhythm in mammals is a daily variation of 24 h that regulates basic physiological processes in such animals [7]. It is produced by a complex network of neurons forming the so-called suprachiasmatic nucleus (SCN) [8]. As recently shown, this oscillator network undergoes spontaneous synchronization in the absence of any environmental input, but its intrinsic synchronization period is then significantly longer than 24 h [9]. Therefore, the actual shorter rhythm results from the environmental entrainment and must be externally imposed. The entrainment is mediated by direct photic inputs from eyes into the SCN, which undergo periodic daily variation. However, it is known that only a distinct subset of neurons in this network is directly influenced by photic inputs [10]. Hence, functioning of this particular neural system is crucially dependent on the ability of the entire complex network to become entrained by an external pacemaker. Analogous behavior can also be expected, for example, in heterogeneous arrays of globally coupled electrochemical oscillators where synchronization and entrainment have been experimentally demonstrated [11].

To understand the operation of pacemakers in networks with complex connection topologies, the action of a pacemaker in a random oscillator network should first be in-

vestigated. In this Letter, networks of identical phase oscillators with random connections are considered. A pacemaker is introduced as a special element whose oscillations have a higher frequency and are not influenced by the rest of the system. Depending on the pacemaker frequency and the strength of coupling, the pacemaker can entrain the entire network, so that the frequencies of all its elements become equal to that of the pacemaker. We find that the entrainment window decreases exponentially with the depth of a network, defined as the mean forward distance of its elements from a pacemaker, and thus only shallow networks can effectively be entrained. This result is confirmed in numerical simulations for several different classes of random networks, including small-world graphs. It is further analytically derived as an approximation for random networks with asymmetric connections.

We consider a system of $N + 1$ phase oscillators, one of them being a pacemaker. The model is given by a set of evolution equations [12] for the oscillator phases ϕ_i and the pacemaker phase ϕ_0,

$$\dot{\phi}_i = \omega - \frac{\kappa}{pN} \sum_{j=1}^{N} A_{ij} \sin(\phi_i - \phi_j) - \mu B_i \sin(\phi_i - \phi_0),$$

$$\dot{\phi}_0 = \omega + \Delta\omega. \tag{1}$$

The topology of network connections is determined by the adjacency matrix **A** whose elements A_{ij} are either 1 or 0. The element with $i = 0$ is special and represents a pacemaker. Its frequency is increased by $\Delta\omega$ with respect to the frequency ω of all other oscillators [13]. The pacemaker is acting on a randomly chosen subset of N_1 elements, specified by B_i taking values 1 or 0. The total number of connections to the pacemaker, $N_1 = \sum_i B_i$, is fixed. The coupling between elements inside the network is characterized by strength κ. The strength of coupling from the pacemaker to the network elements is determined by the parameter μ. In absence of a pacemaker, such networks undergo autonomous phase synchronization at the natural frequency ω. Without loss of generality, we put $\omega = 0$. Moreover, we rescale time as $t' = t\Delta\omega$ and introduce

rescaled coupling strengths $\kappa' = \kappa/\Delta\omega$ and $\mu' = \mu/\Delta\omega$. After such rescaling, the model takes the form of Eq. (1) with $\Delta\omega = 1$ and $\omega = 0$ (we drop primes in the notations for the rescaled couplings). In terms of the original model (1), increasing the rescaled coupling between the elements is equivalent either to an increase of coupling κ or to a decrease of the relative pacemaker frequency $\Delta\omega$.

The presence of a pacemaker imposes hierarchical organization. For any node i, its distance h with respect to the pacemaker is given by the length of the minimum forward path separating this node from the pacemaker. All N_1 elements in the group directly connected to the pacemaker have distances $h = 1$, the next elements that are connected to the elements from this group have distances $h = 2$, etc. Thus, the whole network is divided into a set of shells [14], each characterized by a certain forward distance h from the pacemaker. The set of numbers N_h is an important property of a network. The depth L of a given network, which is the mean distance from the pacemaker to the entire network, is introduced as $L = (1/N)\sum_h h N_h$. It should be noticed that such an ordering of network nodes is based solely on the forward connections descending down the hierarchy and does not depend on the distribution of reverse (upward) connections in the system.

First, we investigated *standard random asymmetric networks*, where independently for all connections $A_{ij} = 1$ with probability p and $A_{ij} = 0$ otherwise. Only sparse random networks with relatively low mean connectivity p and a small number N_1 of elements directly connected to the pacemaker were considered. Numerical simulations were performed for the networks of size $N = 100$ starting with random initial conditions for the phases of all oscillators. For each oscillator, its effective long-time frequency ω_i was computed as $\omega_i = T^{-1}[\phi_i(t_0 + T) - \phi_i(t_0)]$ with sufficiently large T and t_0. The simulations show that the response of a network to the introduction of a pacemaker depends on the strength κ of coupling between the oscillators. When this coupling is sufficiently large (and coupling μ to the pacemaker is also sufficiently strong as assumed below), the pacemaker entrains the whole network (i.e., $\omega_i = 1$ for all elements i). The frozen relative phases $\psi_i \equiv \phi_i - \phi_0$ are displayed in Fig. 1. Here, the elements are sorted according to their hierarchical shells. Despite random variations, there is a clear correlation between phases of oscillators and their positions in the hierarchy. Generally, the phase decreases for deeper shells, and the phase difference between the neighboring shells rapidly becomes smaller as deeper shells are considered. As the coupling strength κ is decreased, the entrainment breaks down at a certain threshold value κ_{cr}. Our simulations show that synchronization between the first and the second shells was almost always the first to break down, and the frequencies of the second and deeper shells remained equal in most cases for the considered random networks.

Figure 2 displays in the logarithmic scale the thresholds κ_{cr} for a large set of networks with different depths and
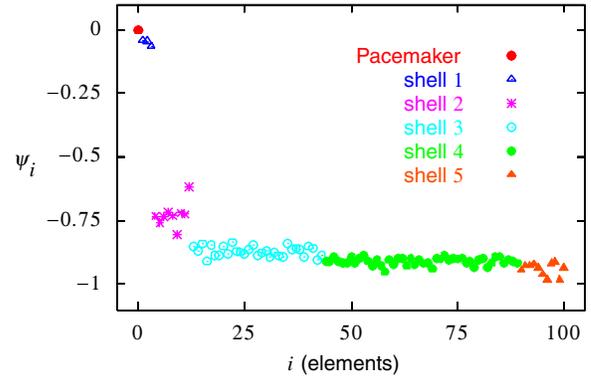

FIG. 1 (color online).   Phases of elements in the entrained state; $N = 100$, $p = 0.05$, $N_1 = 3$, $\kappa = 100$, and $\mu = 1000$.

different numbers of elements in the first shell. Each group with a certain N_1 is displayed by using its own symbol. Every such group generates a cluster of data points. Correlation between the entrainment threshold and the network depth is apparent. The distributions inside each cluster and the accumulation of the clusters yield the dependence $\kappa_{cr}(L)$ of the entrainment threshold on the network depth. Note that the statistical variation of the data becomes larger for deeper networks with larger L and for smaller pN. Similar dependence was found for the networks with different mean connectivity p (see inset). Remarkably, the observed dependences could be well approximated numerically by the exponential dependence

$$\kappa_{cr} \propto (1 + pN)^L. \qquad (2)$$

As the second class, *asymmetric small-world networks* [15] were considered. To generate them, we first constructed a one-dimensional lattice of N elements, where each element had incoming connections from up to its kth neighbor (the degree was thus $2k$). Then, a randomly chosen link in the lattice was eliminated and a distant
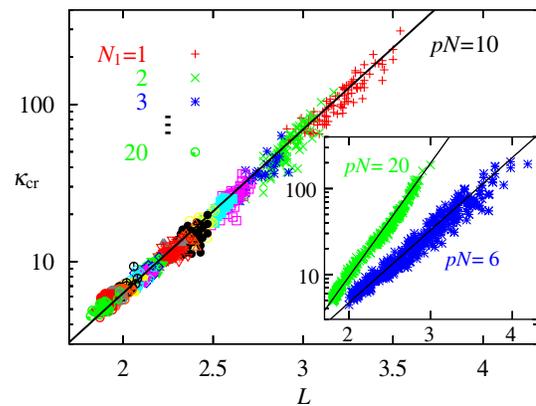

FIG. 2 (color online).   Dependence of the entrainment threshold on the depth L for an ensemble of random networks with $N = 100$ and $p = 0.1$. In the inset, respective data for networks with $p = 0.06$ and 0.2 are plotted. Solid lines are the exponential functions $c_p(1 + pN)^L$ with appropriate fitting parameters c_p.

connection between two independently randomly chosen elements was introduced. This construction was repeated qN times, with the parameter q specifying the randomness of a network. After that, N_1 nodes of the network were randomly chosen and connected to one additional node representing the pacemaker. When q was small, the network was close to a lattice and, in this case, we have seen that stable wave solutions with different winding numbers were possible, depending on initial conditions (cf. [12,16]). To avoid this, we chose almost synchronized states as initial conditions. The entrainment thresholds for such small-world networks are displayed in Fig. 3 and again show a clear correlation between κ_{cr} and L. The dependence on the depth is approximately linear in lattices ($q = 0$), but it becomes strongly nonlinear even when small randomness is introduced. For $q = 0.1$, the dependence is already approximately exponential, though the dispersion of data is strong. As randomness q is increased, the dependence approaches that of the standard random networks with $pN = 2k$.

We have also investigated *asymmetric scale-free random networks* [17], *asymmetric regular random networks* (where every element has exactly the same number of either incoming or outgoing connections), and *symmetric standard random networks*. For all of them, approximately exponential dependences of the entrainment threshold on the network depth were observed in a large parameter region.

The exponential dependence (2) can be approximately derived for asymmetric random networks with large N and pN. In the large-size limit, random graphs have locally a treelike structure [14]. The global tree approximation has previously been used for determining statistical properties of random networks [17]. We apply here the same approximation and assume that the graph of forward connections

FIG. 3 (color online). Dependence of the entrainment threshold on the depth L for an ensemble of small-world networks with $N = 100$, $k = 3$, various N_1, $1 \le N_1 \le 20$, and different randomness q. The solid line is the same exponential function as that fitted to the data with $pN = 6(= 2k)$ in Fig. 2. The dotted line is linear fitting for the regular lattice ($q = 0$).

extending from the pacemaker node represents a tree, so that any oscillator has only one incoming connection from the previous shell. Then the shell populations N_h are given by $N_h = N_1(pN)^{h-1}$ for $h = 2, \ldots, H$, where H is the total number of shells determined by $\sum_{h=1}^{H} N_h = N$. Because pN is large, we have $N_h \ll N_H \simeq N$ for $h < H$, and thus $L \simeq H$. Next, we estimate the numbers m_{hk} of incoming connections leading from all elements in the kth shell to an oscillator in the hth shell. By definition of hierarchical shells, $m_{hk} = 0$ if $k < h - 1$. In the tree approximation, $m_{hk} = 1$ for $k = h - 1$. Because most of the population is concentrated in the last shell, reverse connections from other shells can be neglected. On average, the number of reverse connections from the shell H to an oscillator in the shell h is $m_{hH} = pN_H$. Moreover, the relative statistical deviation from this average is of order $(pN)^{-1/2}$ and is thus negligible. Therefore, in this approximation, all oscillators inside a particular shell have effectively the same number of connections from other shells, and a state with phase synchronization inside each shell is possible. In this state, all oscillators inside a shell have the same phase, i.e., $\phi_i = \theta_h$ for all oscillators i in a shell h. Under entrainment, the phases of such a state can be found analytically as a solution of algebraic equations

$$-\frac{\kappa}{pN} \sum_{k=h-1}^{H} m_{hk} \sin(\theta_h - \theta_k) = 1 \quad \text{for} \quad h = 2, \ldots, n,$$

$$-\mu \sin(\theta_1 - \theta_0) - \frac{\kappa}{pN} \sum_{k=2}^{H} m_{1k} \sin(\theta_1 - \theta_k) = 1, \quad (3)$$

where $\theta_0 \equiv \phi_0$. For large pN, we can linearize $\sin(\theta_h - \theta_k)$ for $h, k \geq 2$ in the solution of Eqs. (3) [it can be shown that $\theta_2 - \theta_H$ is of order $O(1/pN)$]. Furthermore, using $N_H \simeq N \gg N_h$ for $h < H$ and $L \simeq H$, for $h \geq 2$ we obtain

$$\sin(\theta_{h-1} - \theta_h) = \frac{pN}{\kappa}(1 + pN)^{L-h}. \quad (4)$$

Eq. (4) determines the phases of oscillators in the considered synchronized state. Note that the explicit value of the phase θ_1 in this state is not needed below.

The entrainment breakdown can, in principle, occur through destabilization of the synchronized state. Though the analytical proof of its stability is not yet available, our numerical simulations show that the synchronous entrained state with $0 < \theta_{h-1} - \theta_h < \pi/2$ is always stable when it exists. Thus, the breakdown of entrainment in the considered system takes place in a saddle-node bifurcation, through the disappearance of solutions of Eqs. (3). This occurs when $|\sin(\theta_{h-1} - \theta_h)| = 1$ for certain h. For large enough μ, we always have $0 < \sin(\theta_0 - \theta_1) < 1$ (a sufficient condition is $\mu > 1 + \kappa$). Among the other terms, the term $\sin(\theta_1 - \theta_2)$ is always the largest one. Therefore, the solution disappears and breakdown occurs when $\sin(\theta_1 - \theta_2) = 1$. Substituting Eq. (4) into this equation and solving it with respect to κ, we finally derive the dependence (2). Thus, we see that the entrainment breakdown occurs through the loss of frequency locking between
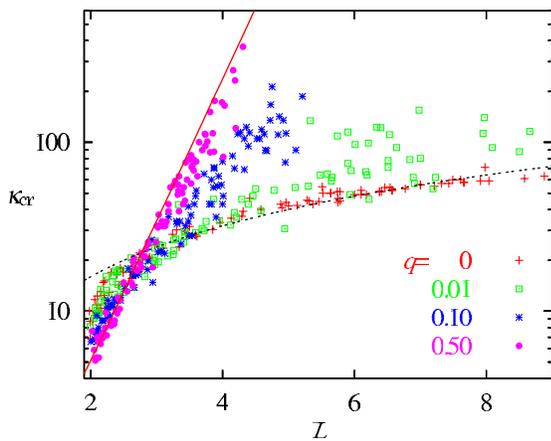
the first shell and the rest of the network. As seen in Fig. 2, the analytical dependence for the critical coupling strength, obtained using the tree approximation, agrees well with the numerical dependence, even for the networks which are not very large.

So far we have used the coupling strength which was rescaled as $\kappa \to \kappa/\Delta\omega$. Therefore, if the nonscaled coupling strength is fixed, Eq. (2) determines the maximum $\Delta\omega_c$ at which the entrainment is still possible, $\Delta\omega_c \propto \kappa(1 + pN)^{-L}$. The entrainment by a pacemaker can take place only if its frequency lies inside the interval $(\omega, \omega + \Delta\omega_c)$.

Thus, the entrainment window *decreases exponentially* with the depth of a network. This is the principal result of our study, which holds not only for standard random networks, where the above analytical estimate is available, but also for small-world graphs and other numerically investigated random topologies. In practice, it implies that only shallow random networks with small depths are susceptible to frequency entrainment.

Our results remain valid when, instead of a pacemaker, external periodic forcing acts on a subset of elements. We have checked that the reported strong dependence on the network depth remains valid for systems with larger network sizes, heterogeneity in frequencies of individual oscillators, and several other coupling functions. The study was performed for coupled phase oscillators which serve as an approximation for various real oscillator systems, including neural networks (see, e.g., [12,18,19]). Its conclusions should be applicable for a broad class of oscillator networks with random architectures.

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[1] A. N. Zaikin and A. M. Zhabotinsky, Nature (London) **225**, 535 (1970).

[2] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980).

[3] K. J. Lee, E. C. Cox, and R. E. Goldstein, Phys. Rev. Lett. **76**, 1174 (1996).

[4] B. Blasius, A. Huppert, and L. Stone, Nature (London) **399**, 354 (1999).

[5] J. J. Tyson and P. C. Fife, J. Chem. Phys. **73**, 2224 (1980); M. Stich and A. S. Mikhailov, Z. Phys. Chem. (Frankfort am Main) **216**, 521 (2002).

[6] See, e.g., M. Stich, M. Ipsen, and A. S. Mikhailov, Phys. Rev. Lett. **86**, 4406 (2001); Physica (Amsterdam) **171D**, 19 (2002).

[7] R. Y. Moore, Ann. Rev. Med. **48**, 253 (1997).

[8] E. E. Abrahamson and R. Y. Moore, Brain Res. **916**, 172 (2001).

[9] S. Yamaguchi *et al.*, Science **302**, 1408 (2003).

[10] S. J. Kuhlman *et al.*, J. Neurosci. **23**, 1441 (2003).

[11] I. Z. Kiss, Y. M. Zhai, and J. L. Hudson, Science **296**, 1676 (2002).

[12] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, New York, 1984).

[13] Note that the system (1) is invariant under transformation $\omega \to -\omega$, $\Delta\omega \to -\Delta\omega$, $\phi \to -\phi$, and therefore the same entrainment behavior takes place when $\Delta\omega < 0$.

[14] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Nucl. Phys. **B653**, 307 (2003); S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks: From Biological Nets to the Internet and WWW* (Oxford University Press, Oxford, 2003).

[15] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).

[16] S. C. Manrubia, A. S. Mikhailov, and D. H. Zanette, *Emergence of Dynamical Order: Synchronization Phenomena in Complex Systems* (World Scientific, Singapore, 2004).

[17] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).

[18] Y. Kuramoto, Physica (Amsterdam) **50D**, 15 (1991).

[19] H. Kori and Y. Kuramoto, Phys. Rev. E **63**, 046214 (2001); H. Kori, Phys. Rev. E **68**, 021919 (2003).