Thermal-Noise Limit in the Frequency Stabilization of Lasers with Rigid Cavities

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We evaluate thermal noise (Brownian motion) in a rigid reference cavity used for frequency stabilization of lasers, based on the mechanical loss of cavity materials and the numerical analysis of the mirror-spacer mechanics with the direct application of the fluctuation dissipation theorem. This noise sets a fundamental limit for the frequency stability achieved with a rigid frequency-reference cavity of order $1\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$ ($0.01\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$) at 10 mHz (100 Hz) at room temperature. This level coincides with the world-highest level stabilization results.

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Introduction.—Thermal fluctuations are fundamental phenomena in physics. Based on statistical physics, Callen and Greene related the spectrum of random motion to loss in a system, obtaining the fluctuation dissipation theorem (FDT) [1]. Although this motion is usually very small in mechanics, especially off resonance, it becomes a serious noise in recent ultraprecise measurements, such as ground-based gravitational wave detection with laser interferometers [2].

On the other hand, an ultrastable (narrow linewidth) laser itself is important for optical frequency standards, high-resolution spectroscopies, and fundamental physics tests as well as interferometric measurements. Therefore, laser frequency stabilization using rigid Fabry-Perot (FP) cavities as the wavelength (frequency) reference has been intensively studied by many groups since an experiment in the 1980s [3]. This kind of ultrastable laser with an FP cavity is planned for future space missions, including the Laser Interferometer Space Antenna (LISA) [4].

Frequency stability achieved with these FP cavities is dependent on the stability of cavity length. However, at nonzero temperatures, the rigid reference cavity has an inevitable mechanical thermal fluctuation which fundamentally limits the frequency stability, even if the coefficient of thermal expansion (CTE) is zero. Nevertheless, as far as the authors know, an estimation of cavity thermal fluctuations has never been rigorously evaluated.

In this Letter, we evaluated thermal fluctuations in a rigid frequency-reference FP cavity. Measured mechanical losses of low CTE materials were used to calculate the frequency stability determined by the cavity's thermal fluctuation associated with these losses. The mechanical equation of motion was solved numerically and the FDT was applied directly to get the thermal-noise spectrum. The result shows that, for widely used materials, the thermal-noise limitation is on the order of $0.1\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$ at 1 Hz, coinciding with the current world-highest level frequency stability with a rigid cavity. This Letter gives new insights on designing ultrastable cavities for the precision measurement community.

Experiment.—A basic quantity in evaluating thermal noise is mechanical loss, ϕ, of the cavity materials used as the spacer, mirror substrate, and reflective coating. The spacer materials, in particular, are usually chosen from the viewpoint of low CTE to minimize the coupling of temperature variations to the optical path length and frequency stability. The mirror substrate material is often similarly chosen. However, a lower emphasis has been placed on the loss of these materials. Thus, we measured the mechanical quality factor, Q $(= 1/\phi)$, of several low CTE materials, including ULE and Zerodur. Each sample had a cylindrical shape [similar to Fig. 1(d)]. To get Qs we performed a ring-down measurement, which measures a decay time, $\tau\ [= Q/(\pi f_0)]$, of free mechanical oscillations at the sample resonances (frequency f_0).

In the measured frequency range, from a few kHz to 100 kHz, a frequency dependence of loss was not observed. Therefore, the structural damping model [5], which assumes constant ϕ at all frequencies, was adopted in the following arguments. Among low CTE materials, ULE has a relatively high Q (6.1×10^4) compared to Zerodur (3.1×10^3). However, this value is lower than fused silica (typically constant $Q \sim 10^6$ [6]), which is used for optics when thermal noise is an important consideration.

Order estimation.—In the following, we discuss the calculation of thermal fluctuation in an FP cavity where

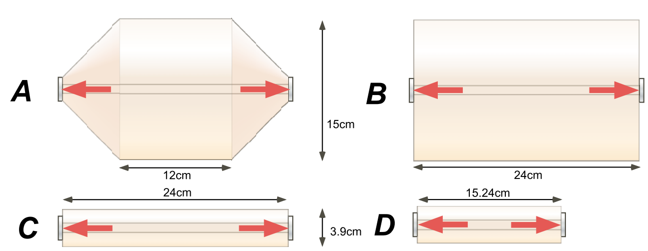

FIG. 1 (color online). Assumed cavity shapes. Common parameters are axial hole diameter: 10 mm, mirror diameter: 25.4 mm, mirror thickness: 5 mm, and coating thickness: 2 μm. The thick arrows show positions of Gaussian forces in the calculation.

two mirrors are optically contacted to the ends of a rigid spacer. To get a rough order estimate, we calculated thermal fluctuation in the spacer and mirror with coating separately and then added the results, assuming that every noise component is uncorrelated. Although this is not a rigorous treatment, the process may help our understanding.

The FDT is a unique way to obtain the thermal fluctuation spectrum. The basic form of the FDT is [1,5]

$$G_x(f) = -\frac{4k_B T}{\omega}\mathrm{Im}[H(\omega)]. \quad (1)$$

Here, $G_x(f)$ is the one-sided power spectrum of the displacement x at frequency f, k_B is the Boltzmann constant, T is the temperature, and ω is the angular frequency. $H(\omega)$ is the transfer function from force $f(t)$ applied to the observing point (or area) to displacement $x(t)$ defined as $H(\omega) \equiv \tilde{X}(\omega)/\tilde{F}(\omega)$, where $\tilde{X}(\omega)$ and $\tilde{F}(\omega)$ are the Fourier transforms of $x(t)$ and $f(t)$, respectively. $\mathrm{Im}[H(\omega)]$ is proportional to the loss of the system. The FDT has a wide range of experimental validity in mechanics, including ~mHz [7,8], ~100 Hz [9], and up to 100 kHz measurements [10].

Using this theorem, we first estimate the contribution from the spacer, regarding the mirrors as small accompaniments of the spacer. The system is assumed to be a cylindrical elastic bar. By considering the low-frequency portion of $H(\omega)$ [11], the longitudinal thermal fluctuation of one end is written, using the spacer's loss, ϕ_{spacer}, as

$$G_{\mathrm{spacer}}(f) = \frac{4k_B T}{\omega}\frac{L}{3\pi R^2 E}\phi_{\mathrm{spacer}}. \quad (2)$$

Here, R is the radius, L is the length, and E is Young's modulus. Assuming $T = 300$ K, $R = 4$ cm, $L = 24$ cm, $\phi_{\mathrm{spacer}} = 1/(6\times 10^4)$, and $E = 6.8\times 10^{10}$ Pa (ULE values), the contribution from one end of the spacer is $\sqrt{G_{\mathrm{spacer}}} = 3.2\times 10^{-18}$ m/$\sqrt{\mathrm{Hz}}$ at 1 Hz.

Next, we evaluate the contribution from the mirror. A simple analytic expression of $\mathrm{Im}[H(\omega)]$ for a mirror is obtained only when we regard the mirror body as an infinite-half volume. This is a good approximation when the mirror substrate is sufficiently larger than the Gaussian beam. The analytic result shows that the mirror fluctuation, G_{mirror}, is written as [12–14]

$$G_{\mathrm{mirror}}(f) = \frac{4k_B T}{\omega}\frac{1-\sigma^2}{\sqrt{\pi}E w_0}\phi_{\mathrm{sub}}. \quad (3)$$

Here, σ is Poisson's ratio, w_0 is the beam radius, and ϕ_{sub} is the loss of the mirror substrate. When the beam radius decreases, the thermal noise increases because canceling happens only in fluctuations on a smaller scale than the beam radius. By substituting $w_0 = 240$ μm, $\phi_{\mathrm{sub}} = 1/(6\times 10^4)$, $E = 6.8\times 10^{10}$ Pa, and $\sigma = 0.18$ (ULE), we get $\sqrt{G_{\mathrm{mirror}}} = 3.8\times 10^{-17}$ m/$\sqrt{\mathrm{Hz}}$ at 1 Hz.

In reality, because the mirror's reflective coating introduces additional losses, G_{mirror} must be multiplied by the following correction factor [14,15]:

$$\sim\left(1 + \frac{2}{\sqrt{\pi}}\frac{1-2\sigma}{1-\sigma}\frac{\phi_{\mathrm{coat}}}{\phi_{\mathrm{sub}}}\frac{d}{w_0}\right). \quad (4)$$

Here, d is the coating thickness and ϕ_{coat} is its loss. This expression weights the ratio of the coating and substrate losses and the ratio of coating thickness and beam radius. Assuming $d = 2$ μm and $\phi_{\mathrm{coat}} = 4\times 10^{-4}$ (typically measured between 100 Hz and 10 kHz [15–17]), $\sqrt{G_{\mathrm{mirror}}}$ is then 4.2×10^{-17} m/$\sqrt{\mathrm{Hz}}$.

If we add contributions from two mirrors and two ends of the spacer, assuming everything is uncorrelated, measuring no common mode cancellations and assuming no coherence enhancements in differential modes, the total length fluctuation, $\sqrt{G_L}$, becomes 5.9×10^{-17} m/$\sqrt{\mathrm{Hz}}$ at 1 Hz with a frequency dependence of $f^{-1/2}$. The spacer, mirror substrate, and coating provide 1%, 84%, and 15% of G_L, respectively. This displacement fluctuation results in a frequency noise of 0.13 Hz/$\sqrt{\mathrm{Hz}}$ with 563 nm light and a cavity length of 24 cm because $\sqrt{G_L}/L = \sqrt{G_\nu}/\nu$. Here, $\sqrt{G_\nu}$ is the frequency noise and ν is the light frequency. This rough estimate is a good approximation of the world-highest level stabilization result using a cavity (shown later).

Calculation method.—To calculate an accurate thermal-noise level, we must take everything into account simultaneously and precisely—the two mirrors are finite sized and optically contacted onto the spacer, which is a three-dimensional cylinder. The differential distance between them is measured with a Gaussian weighting.

The most concise way to solve this kind of mixture problem is numerically. Because the modal expansion method is not applicable for an accurate calculation [18], we use the FDT directly. Levin proposed a more useful form of the FDT for this "direct approach" [13]:

$$G_X(f) = \frac{8k_B T}{\omega^2}\frac{W_{\mathrm{diss}}(f)}{F_0^2}. \quad (5)$$

Here, G_X is the power spectrum of an observable physical quantity X, and W_{diss} is the average dissipated energy when the force $F_0\cos(2\pi f t)\boldsymbol{P}(\mathbf{r})$ is applied to the system. (This is the conjugate force to X.) Using a weighting function, $\boldsymbol{P}(\mathbf{r})$, we can write $X(t) = \int \boldsymbol{u}(\mathbf{r},t)\cdot\boldsymbol{P}(\mathbf{r})dV$. Here, $\boldsymbol{u}(\mathbf{r},t)$ is the displacement of the system at position $\mathbf{r}$.

Therefore, to obtain the cavity's thermal-noise spectrum, G_X, we calculate the dissipated energy by applying cyclic forces with Gaussian profiles, $\boldsymbol{P}$, at the required frequency, f, onto mirrors connected by a spacer. W_{diss} is proportional to both the loss, $\phi(\mathbf{r}, f)$, and strain energy at position $\mathbf{r}$. The strain energy is obtained by solving the equation of motion for the entire system. To get the differential displacement between beam-illuminated areas, we apply two opposing Gaussian forces simultaneously [19]. This procedure intrinsically includes the loss distribution and correlation between the two areas.

Result.—For the numerical modeling, we chose typical straight FP cavities—from an oblong, thick cylinder to a thin, compact cylinder (Fig. 1). We varied the shape of the cavities, the materials of the spacer and mirror substrate, and the beam radius on the mirror surface. We used ANSYS, a commercial program, to solve the problem with the finite element method.

Table I summarizes eight cases of our calculation results, showing the displacement noise level at 1 Hz along with contributions from each component. Corresponding frequency stability limits are also shown, assuming 563 nm light. These results suggest that the thermal-noise limitation is about $0.1 \times (1\ \mathrm{Hz}/f)^{1/2}\mathrm{Hz}/\sqrt{\mathrm{Hz}}$, although the selection of materials, dimensions, and/or beam radius may nominally change this result.

In every case, the mirror is the dominant thermal-noise source. This is because, at a frequency region well below the mechanical resonance, only the losses around the beam spot contribute to thermal noise. Therefore, the cavity's overall shape and/or the loss of the spacer do not greatly affect thermal noise when other conditions around the mirror, such as the beam radius, are kept identical (case No. 2/6/7 and case No. 2/3). Because the mirror contribution is dominant, noise decreases with a larger beam radius or with a lower-loss mirror substrate (e.g., case No. 2/4 and case No. 1/2). In the case of the fused silica mirror substrate, the contribution from the coating becomes the most dominant source (e.g., case No. 8). Because we cannot greatly alter the coating thickness, the coating loss, or the beam radius, thermal noise from the coating becomes the practical limitation.

Comparison with experiment.—We compared our calculation result with the world-best frequency stability below 1 Hz achieved with a rigid cavity, as reported by a National Institutes of Standards and Technology (NIST) group [20]. Their cavity geometry and materials are similar to our case No. 1. At a gate time, τ, between 1 and 300 sec, the Allan variance [21], $\sigma_y(\tau)$, a measure of the fractional frequency stability, is similar to the average value of $\sim 4 \times 10^{-16}$. It is converted into frequency noise of $\sim 0.2 \times (1\ \mathrm{Hz}/f)^{1/2}\mathrm{Hz}/\sqrt{\mathrm{Hz}}$ between 3 mHz and 1 Hz with 563 nm light, assuming a mathematical relationship of $\sqrt{G(f)} = \sigma_y \nu/\sqrt{2(\ln 2) f}$ (valid for constant σ_y). Figure 2 shows a comparison of our calculation and experimental results. The calculation coincides with experimental observation. Therefore, the measured stability is seemingly limited by the cavity thermal noise. A lower stability was found with the use of the same cavity shape with spacer and mirrors made of Zerodur [23]. This is expected from our calculation (case No. 5)—the lower Q of Zerodur increases the internal-loss-induced thermal fluctuation.

Our results also hold at frequencies above 1 Hz and suggest that there is a limiting thermal-noise wall around $0.01\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$ at 100 Hz. This coincides with the results describing the frequency stability of a rigid cavity used for the ground-based gravitational wave detector VIRGO [22]. This is the highest known stability result above 1 Hz. In this experiment, a triangular ring cavity with mirrors and spacer [similar to Fig. 1(a)] made of ULE was used. We also compare the VIRGO data with our calculation for case No. 1 in Fig. 2, since the mirror substrate, which is the dominant source of thermal noise, is the same for both the NIST and VIRGO cavities [24].

Discussions.—For the NIST and VIRGO cavities, our calculation predicts that, if their mirrors were made of silica, the stability would be further improved by a factor of ~ 2 (case No. 2). If the beam radius were enlarged, a total factor of ~ 3 improvement is expected (case No. 4). The reduction of loss in coating would then become more relevant for this work [16,17]. Cooling the cavity will also help, assuming the FDT temperature dependence, and assuming material losses which decrease with temperature (*not* true for fused silica [25]).

We assumed a constant loss, ϕ, against frequency, resulting in a thermal-noise increase with an $f^{-1/2}$ dependence toward lower frequencies. This assumption for the mirror substrate and coating at the $\phi \sim 10^{-4}$ level has been proven by the direct measurement of the $f^{-1/2}$ thermal noise in an FP cavity between 100 Hz and 100 kHz [10]. The measured thermal noise in a torsion pendulum also suggests that a constant ϕ of $\sim 10^{-4}$ is still reasonable between 0.01 and 10 mHz [8], although ϕ must converge to

TABLE I. Calculation results. Displacement and frequency noise values are shown at 1 Hz. Underlines represent the dominant noise contribution for G in parts percentage. Assumed losses, ϕ, for ULE, fused silica, Zerodur, and coating are $1/(6 \times 10^4)$, $1/(1 \times 10^6)$, $1/(3 \times 10^3)$, and 4×10^{-4}, respectively. "Substrate" means mirror substrate.

Case No.	Shape (Fig. 1)	Spacer material	Substrate material	Beam radius w_0 (μm)	Displacement noise at 1 Hz ($\times 10^{-17}$) (m/$\sqrt{\mathrm{Hz}}$)	Contribution in G (%) Spacer	Substrate	Coating	Frequency noise at 1 Hz (Hz/$\sqrt{\mathrm{Hz}}$) for 563 nm light
1	A	ULE	ULE	240	6.0	3	<u>84</u>	13	0.13
2	A	ULE	Silica	240	2.6	14	24	<u>62</u>	0.059
3	A	Silica	Silica	240	2.5	1	28	<u>71</u>	0.055
4	A	ULE	Silica	370	2.0	25	28	<u>47</u>	0.044
5	A	Zerodur	Zerodur	240	21	3	<u>96</u>	1	0.47
6	B	ULE	Silica	240	2.6	14	24	<u>62</u>	0.058
7	C	ULE	Silica	240	2.8	25	21	<u>54</u>	0.063
8	D	ULE	Silica	200	3.1	16	21	<u>63</u>	0.11

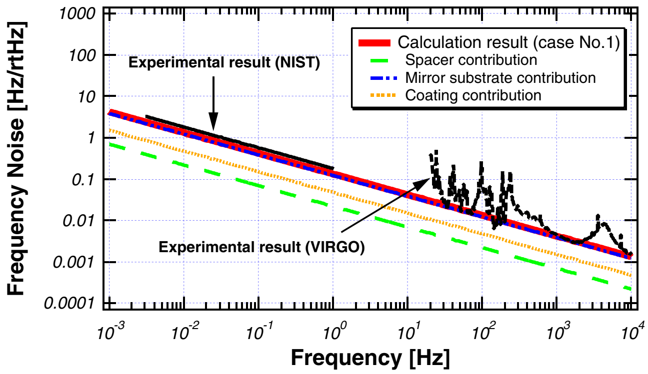

FIG. 2 (color online). Calculation result (case No. 1) and the experimental results (converted from Refs. [20,22] into frequency noises at 563 nm). In this case, the contribution from the mirror substrate is dominant, because it is made of ULE. The experimental results agree well with our calculation. The discrete calculation performed at frequency intervals of $\sim$10 Hz was interpolated to give the expected smooth $f^{-1/2}$ dependence in this plot.

zero at zero frequency due to mathematical requirement. Therefore, it is reasonable to assume constant ϕ of cavity materials at wider frequency ranges, based on experimental observation at high frequencies. Recent reports suggest, however, that low-loss fused silica ($\phi \lesssim 10^{-7}$) and coating may have a frequency dependent ϕ [17,26]. Also, below $\sim$100 Hz, less experimental clues support the model. Thus, the frequency behavior and an absolute value for loss of the cavity material should be more carefully verified by experimentally measuring the loss at the required frequency for frequency stabilization.

Although only intrinsic losses in the cavity materials were considered as thermal-noise sources, optical contacts and surfaces must also be assessed. We evaluated these effects by calculation. If the products of effective loss, ϕ, and thickness, h, of the contact area or the surfaces are smaller than 10^{-6} m, we will see less than a 10% increase in our result shown in Table I. Although we believe this to be true, experimental verification is important. We also neglected support losses because the loss in the spacer has the lowest contribution to the thermal-noise level. Finally, thermoelastic losses [27] cause "thermoelastic noise," which is interpreted as the direct coupling between CTE and statistical temperature fluctuation within a specific volume [12]. For this analysis, the noise is considered to be small when the mirrors and spacers are made of fused silica and/or ULE [28].

Conclusion.—Thermal fluctuation becomes an inevitable noise source in precise experiments. We rigorously evaluated the thermal noise from a rigid cavity based on our mechanical loss measurement and direct numerical analysis of the system with the FDT. The noise, originating mainly from the cavity mirrors, lies around $0.1 \times (1\ \mathrm{Hz}/f)^{1/2} \mathrm{Hz}/\sqrt{\mathrm{Hz}}$ at room temperature with a typical setup. Comparing this with past experimental results, we found that thermal motion fundamentally limits the linewidth and could explain the world-highest level frequency stabilization results. We suggested that using low-loss mirrors and a larger spot size may improve the frequency stability. The results revealed that thermal noise from cavity components, as well as temperature stability and a small CTE, must be taken into account when designing ultrastable cavities. We believe that our results will help experimentalists who use frequency-stabilized lasers realize the importance of cavity thermal noise.

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