Nearly Deterministic Linear Optical Controlled-NOT Gate

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We show how to construct a near deterministic CNOT gate using several single photons sources, linear optics, photon number resolving quantum nondemolition detectors, and feed forward. This gate does not require the use of massively entangled states common to other implementations and is very efficient on resources with only one ancilla photon required. The key element of this gate is nondemolition detectors that use a weak cross-Kerr nonlinearity effect to conditionally generate a phase shift on a coherent probe if a photon is present in the signal mode. These potential phase shifts can then be measured using highly efficient homodyne detection.

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In the past few years we have seen the emergence of single photon optics with polarization states as a realistic path for achieving universal quantum computation. This started with the pioneering work of Knill, Laflamme, and Milburn (KLM) [1] who showed that with only single photon sources, detectors, and linear elements such as beam splitters, a near deterministic CNOT gate could be created with the use of significant but polynomial resources. With this architecture for the CNOT gate and trivial single qubit rotations, a universal set of gates is hence possible and a route forward for creating large devices can be seen. Since this original work, there has been significant progress both theoretically [2–6] and experimentally [7–9], with a number of CNOT gates actually demonstrated.

Much of the theoretical effort has focused on determining more efficient ways to perform the controlled logic. The standard model for linear logic uses only [1] single photon sources, linear optical elements including feed forward, photon number resolving single photon detectors, and it has been shown by Knill [4] that the maximum probability for achieving the CNOT gate is $3/4$. While these upper bounds are not thought to be tight, with the best success probabilities for the CNOT gate being $2/27$ [10], it does indicate that near deterministic gates are not possible using only the above resources and strategy. These gates can be made efficient using the ''standard'' optical teleportation tricks which require the use of massively entangled resources. Are there other natural ways to increase the efficient of these gate operations? Franson *et al.* [2] showed that if you can increase your allowed physical resources to include maximally entangled two photon states, then the CNOT gate can have its probability of success boosted to $1/4$, though this is still far below the $3/4$ maximum. Alternatively it is possible to use single photons for the cluster state method of one way quantum computation [5,6]. This can dramatically decrease the number of single photons sources required to perform a CNOT gate [from up to 10 000 for KLM logic to on average 45 Bell pairs (90 single photons) for the cluster approaches]. The overhead here in single photon sources is large (but polynomial and hence still efficient in a sense). Can we, however, build near deterministic (or deterministic) linear optics gates with a low overhead for sources and detectors by relaxing the constraints in the standard model?

There are several options here: we can change the way in which we encode our information (from polarization encoded single photon qubits) or the mechanism by which we condition and detect them. There have been schemes by Yoran and Reznik [11] that encode their information in both polarization and which path. This encoding allows a deterministic Bell state measurement but the basic gate operations are still relatively inefficient. Alternatively one could encode the information in coherent states of light as proposed by Ralph *et al.* [12]. A key issue here becomes the creation and detection of superpositions of coherent states. If we want to maintain encoding our information in polarization states of light, what else is possible? The main architecture freedom we have left to change are the single photon detectors. We could move to nondestructive quantum nondemolition detectors (QND) which would have the potential available of being able to condition the evolution of our system but without necessarily destroying the single photons [13–15]. They can also resolve one photon from a superposition of zero and two. QND devices are generally based on cross-Kerr nonlinearities. Historically these reversible nonlinearities have been extremely tiny and unsuitable for single photon interactions but recently giant Kerr nonlinearities have become available with electromagnetically induced transparency (EIT) [16]. It is currently not clear whether these nonlinearities are sufficient from the natural implementation of single photon-single photon quantum gates; however, they can be used for QND detection where we require a single photon-large coherent beam interaction. Here the nonlinearity strength needs to be sufficient only for a small phase shift to be induced onto a coherent probe beam (which is distinguishable from the original probe) [17].

Now that we have decided to use QND detection for linear optical quantum computation we need to investigate its effect on the CNOT gates and this is the key purpose of this Letter. We could investigate each of the known gates in turn but we will focus on the Franson's 3 photon CNOT gate [7], the reason being that it requires fewer physical resources [18]. We will show that a near deterministic CNOT gate can be performed with such QND detectors without destroying the ancilla photon, provided feed forward is available. More generally we will show that for a n qubit circuit, the number of single photon sources requires scales as $n + 1$. The extra photon is, however, not destroyed in the computation and is left at the end. It is not consumed in the computation. This approach can also be applied to achieve cluster state computing or computing by measurement alone [5,6].

Before we begin our detailed discussion, let us first consider the photon number QND measurement using a cross-Kerr nonlinearity, which has a Hamiltonian of the form $H_{\rm QND} = \hbar\chi a_s^\dagger a_s a_p^\dagger a_p$ where the signal (probe) mode has the creation and destruction operators given by $a_s^\dagger$, a_s ($a_p^\dagger$, a_p), respectively, and χ is the strength of the nonlinearity. If we consider the signal state to have the form $|\psi\rangle = c_0|0\rangle_s + c_1|1\rangle_s$ with the probe beam initially in a coherent state $|\alpha\rangle_p$, then the cross-Kerr interaction causes the combined signal-probe system to evolve as

$$U_{ck}|\psi\rangle_s|\alpha\rangle_p = e^{iH_{\rm QND}t/\hbar}[c_0|0\rangle_s + c_1|1\rangle_s]|\alpha\rangle_p = c_0|0\rangle_s|\alpha\rangle_p + c_1|1\rangle_s|\alpha e^{i\theta}\rangle_p, \quad (1)$$

where $\theta = \chi t$ with t being the interaction time. We observe immediately that the Fock state $|n_a\rangle$ is unaffected by the interaction but the coherent state $|\alpha_c\rangle$ picks up a phase shift directly proportional to the number of photons n_a in the $|n_a\rangle$ state. For n_a photons in the signal mode, the probe beam evolves to $|\alpha e^{in_a\theta}\rangle_p$. Assuming $\alpha\theta \gg 1$, a measurement of the phase of the probe beam (via homodyne-heterdyne techniques) projects the signal mode into a definite number state or superposition of number states. The requirement $\alpha\theta \gg 1$ is interesting, as it tells us that a large nonlinearity θ is not absolutely required to distinguish different $|n_a\rangle$, even for zero, one, and two Fock states. We could have θ small but would then require α, the amplitude of the probe beam large. This is entirely possible and means that we can operate in the regime $\theta \ll 1$, which is experimentally more realizable. If this cross-Kerr nonlinearity were going to be used directly to implement a CPHASE-CNOT gate between single photons then we would require $\theta = \pi$.

In this Fock state detection model we measure the phase of the probe beam immediately after it has interacted with the weak cross-Kerr nonlinearity. This is the regime where the QND detector functions like the standard single photon detector. However, if we want to do a more ''generalized'' type of measurement between different signal beams, we could delay the measurement of the probe beam instead, having the probe beam interact with several cross-Kerr nonlinearities where the signal mode is different in each case. The probe beam measurement then occurs after all these interactions in a collective way which could for instance allow a nondestructive detection that distinguishes superpositions and mixtures of the states $|HH\rangle$ and $|VV\rangle$ from $|HV\rangle$ and $|VH\rangle$. The key here is that we could have no net phase shifts on the $|HH\rangle$ and $|VV\rangle$ terms while having a phase shift on the $|HV\rangle$ and $|VH\rangle$ terms. We will call this generalization a *two qubit polarization parity QND detector* and it is this type of detector that allows us to circumvent the Knill bounds.

Consider two polarization qubits initially prepared in the states $|\Psi_1\rangle = c_0|H\rangle_a + c_1|V\rangle_a$ and $|\Psi_2\rangle = d_0|H\rangle_b + d_1|V\rangle_b$. These qubits are split individually on polarizing beam splitters (PBS) into spatial modes which then interact with cross-Kerr nonlinearities as shown in Fig. 1. The action of the PBS's and cross-Kerr nonlinearities evolve the combined system $|\Psi_1\rangle|\Psi_2\rangle|\alpha\rangle_p$ will evolve to $|\psi\rangle_T = [c_0d_0|HH\rangle + c_1d_1|VV\rangle]|\alpha\rangle_p + c_0d_1|HV\rangle|\alpha e^{i\theta}\rangle_p + c_1d_0|VH\rangle|\alpha e^{-i\theta}\rangle_p$. We observe immediately that the $|HH\rangle$ and $|VV\rangle$ pick up no phase shift and remain coherent with respect to each other. The $|HV\rangle$ and $|VH\rangle$ pick up opposite sign phase shift θ which could allow them to be distinguished by a general homodyne-heterodyne measurement. However, if we choose the local oscillator phase $\pi/2$ offset from the probe phase (we will call this an X quadrature measurement), then the states $|\alpha e^{\pm i\theta}\rangle_p$ can not be distinguished [19]. More specifically, with α real, an X homodyne measurement conditions $|\psi\rangle_T$ to

$$|\psi_X\rangle_T = f(X,\alpha)[c_0d_0|HH\rangle + c_1d_1|VV\rangle] + f(X,\alpha\cos\theta)[c_0d_1e^{i\phi(X)}|HV\rangle + c_1d_0e^{-i\phi(X)}|VH\rangle], \quad (2)$$

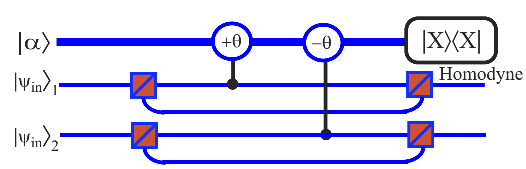

FIG. 1 (color online). Schematic diagram of a two qubit polarization QND detector that distinguishes superpositions and mixtures of the states $|HH\rangle$ and $|VV\rangle$ from $|HV\rangle$ and $|VH\rangle$ using several cross-Kerr nonlinearities and a coherent laser probe beam $|\alpha\rangle$. The scheme works by first splitting each polarization qubit into a which path qubit on a polarizing beam splitter. The action of the first cross-Kerr nonlinearity puts a phase shift θ on the probe beam only if a photon was present in that mode. The second cross-Kerr nonlinearity puts a phase shift $-\theta$ on the probe beam only if a photon was present in that mode. After the nonlinear interactions, the which path qubits are converted back to polarization encoded qubits. The probe beam only picks up a phase shift if the states $|HV\rangle$ and/or $|VH\rangle$ were present and hence the appropriate homodyne measurement allows the states $|HH\rangle$ and $|VV\rangle$ to be distinguished from $|HV\rangle$ and $|VH\rangle$. The two qubit polarization QND detector is a parity checking device.

where $f(x, \beta) = \exp[-\frac{1}{4}(x - 2\beta)^2]/(2\pi)^{1/4}$ and $\phi(X) = \alpha x \sin\theta - \alpha^2 \sin 2\theta (\mathrm{Mod} 2\pi)$. We see that $f(X, \alpha)$ and $f(X, \alpha\cos\theta)$ are two Gaussian curves with the midpoint between the peaks located at $X_0 = \alpha[1 + \cos\theta]$ and the peaks separated by a distance $X_d = 2\alpha[1 - \cos\theta]$. As long as this difference is large $\alpha\theta^2 \gg 1$, then there is little overlap between these curves. Hence for $X > X_0$,

$$|\psi_{X>X_0}\rangle_T \sim c_0 d_0 |HH\rangle + c_1 d_1 |VV\rangle, \tag{3}$$

while for $X < X_0$,

$$|\psi_{X<X_0}\rangle_T \sim c_0 d_1 e^{i\phi(X)} |HV\rangle + c_1 d_0 e^{-i\phi(X)} |VH\rangle. \tag{4}$$

We have used the approximate symbol ($\sim$) in these equations as there is a small but finite probability that the state (3) can occur for $X < X_0$. The probability of this error occurring is given by $P_{\mathrm{error}} = \frac{1}{2}(1 - Erf[X_d/2\sqrt{2}])$ which is less than 10^{-5} when the distance $X_d \sim \alpha\theta^2 > 9$. This shows that it is still possible to operate in the regime of weak cross-Kerr nonlinearities, $\theta \ll \pi$.

The action of this two-mode polarization nondemolition parity detector is now very clear; it splits the even parity terms (3) nearly deterministically from the odd parity cases (4). This is really the power enabled by nondemolition measurements and why we can engineer strong nonlinear interactions using weak cross-Kerr effects. Above we have chosen to call the even parity state $\{|HH\rangle, |VV\rangle\}$ and the odd parity states $\{|HV\rangle, |VH\rangle\}$, but this is an arbitrary choice primarily dependent on the form-type of PBS used to convert the polarization encoded qubits to which path encoded qubits. Any other choice is also acceptable and it does not have to be symmetric between the two qubits.

It is also interesting to look at the $X < X_0$ solution given by (4). We observe immediately that this state is dependent on the measured X homodyne value and hence the state is conditioned dependent on our measurement result X. However, simple local rotations using phase shifters dependent on the measurement result X can be performed via a feed-forward process to transform this state to $c_0 d_1 |H\rangle_a |V\rangle_b + c_1 d_0 |V\rangle_a |H\rangle_b$ which is independent of X. These transformations are very interesting as it seems possible with the appropriate choice of c_0, c_1 and d_0, d_1 to create arbitrary entangled states near deterministically. For instance, if we choose $d_0 = d_1 = 1/\sqrt{2}$, then our device outputs either the state $c_0|HH\rangle + c_1|VV\rangle$ or $c_0|HV\rangle + c_1|VH\rangle$. A simple bit flip on the second polarization qubit transforms it into the first. Thus our two mode parity QND detector can be configured to act as a near deterministic entangler (see Fig. 2). This gate allows us to take two separable polarization qubits and efficiently entangle them (near deterministically). If each of our qubits are initially $|H\rangle + |V\rangle$ then the action of this entangling gate is to create the maximally entangled state $|HH\rangle + |VV\rangle$. Generally it was thought that strong nonlinearities are required to do this near deterministically; however, our scheme here is using only weak nonlinearities $\theta \ll \pi$. This gate is critical and forms the key element for our efficient Franson CNOT gate. It can also obviously be used to generate maximally entangled state required for several of the other CNOT implementations.

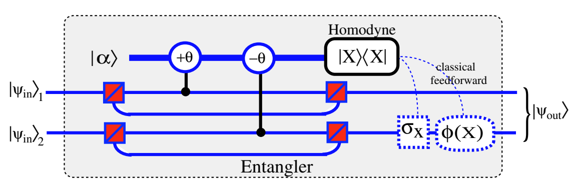

FIG. 2 (color online). Schematic diagram of a two polarization qubit entangling gate. The basis of the scheme uses the QND-based parity detector described in Fig. 1. If we consider that the input state of the two polarization qubit is $|HH\rangle + |HV\rangle + |VH\rangle + |VV\rangle$ then after the parity gate we have conditioned on an X homodyne measurement either the state $|HH\rangle + |VV\rangle$ or $e^{i\phi(X)}|HV\rangle + e^{-i\phi(X)}|VH\rangle$ where $\phi(X)$ is a phase shift dependent on the result of the homodyne measurement. A simple phase shift achieved via classical feed forward then allows this second state to be transformed to the first.

Now let us move our attention to the construction of the CNOT gate (depicted in Fig. 3). This is the analogue of the Franson CNOT gate from [7] but with the key PBS and 45-PBS replaced with $\{H, V\}$ and $\{D = H + V, \bar{D} = H - V\}$ two polarization qubit entangling gates. Franson's photon number resolving detectors have also been replaced with single photon number resolving QND detectors. Consider an initial state of the form $[c_0|H\rangle_c + c_1|V\rangle_c] \otimes [|H\rangle + |V\rangle] \otimes [d_0|H\rangle_t + d_1|V\rangle_t]$. The action of the left-hand side entangler evolves the system to

$$[c_0|HH\rangle + c_1|VV\rangle] \otimes [d_0|H\rangle_t + d_1|V\rangle_t]. \tag{5}$$

Now the action of the 45-entangling gate (where the PBS in the original gate have been replaced with 45-PBS's) transforms the state to $\{c_0|H\rangle - c_1|V\rangle\}(d_0 - d_1)|\bar{D}, \bar{D}\rangle + \{c_0|H\rangle + c_1|V\rangle\}(d_0 + d_1)|D, D\rangle$ where for the $X < X_0$ measurement we have performed the usual phase correction, bit flip, and an addition sign change $|V\rangle \to -|V\rangle$ on the first qubit. The second mode is now split on a normal $\{H, V\}$ PBS and a QND photon number measurement performed. A bit flip is performed if a photon is detected in the V mode. The final state from these interactions and

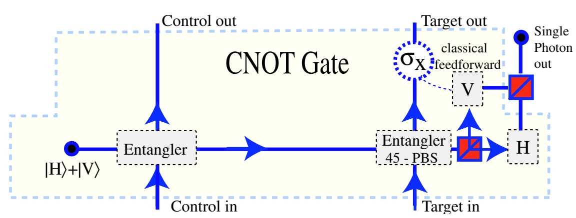

FIG. 3 (color online). Schematic diagram of a near deterministic CNOT composed of two polarization qubit entangling gates (one with PBS in the $\{H, V\}$ basis and one with PBS in the $\{D, \bar{D}\}$ basis), one ancilla signal photon $|H\rangle + |V\rangle$, feed-forward elements, and four single photon resolving QND detectors.

feed-forward operations [20] is

$$c_0 d_0|HH\rangle + c_0 d_1|HV\rangle + c_1 d_0|VV\rangle + c_1 d_1|VH\rangle, \quad (6)$$

which is the same state obtained by performing a CNOT operation on the state $[c_0|H\rangle_c + c_1|V\rangle_c] \otimes [d_0|H\rangle_t + d_1|V\rangle_t]$. This shows that our QND-based gates have performed a near deterministic CNOT operation. The core element of this gate is the *two qubit polarization parity QND detector* which engineers a two polarization qubit interaction via a strong probe beam. At the heart of this detector are weak cross-Kerr nonlinearities that make it possible to distinguish subspaces of basis states from others which is not possible with convenient destructive photon counters. It is this that allows us to exceed the Knill bounds presented in [4]. From a different perceptive our two mode QND entangling gate is acting like a fermonic polarizing beam splitter; that is, it does not allow the photon bunching effects. Without these photon bunching effects simple feed-forward operations allows our overall CNOT gate to be made near deterministic. This represents a huge saving in the physical resources to implement single photon quantum logic. For the CNOT operation, only one extra ancilla photon is needed beyond the control and target photons to perform the gate operation in the near deterministic fashion. In fact, it is straightforward to observe that if we want to do an n qubit computation (with number of one and two qubit gates), only $n + 1$ single photon sources would be required in principle.

The resources required to perform this QND-based CNOT gate as presented here are: three single photon sources (two to encode the control and target qubits and one ancilla), six weak cross-Kerr nonlinearities, two coherent light laser probe beams and homodyne detectors, plus basic linear optics elements to convert polarization encoded qubits to spatial coding ones and perform the feed forward. The single photon ancilla is not consumed in the gate operation and can be recycled for further use. This compares with potentially thousands of single photon sources, detectors, and linear optical elements to implement the original KLM gate. It is possible to construct this near deterministic CNOT with fewer cross-Kerr nonlinearities (potentially as few as two but recycling them) but as a cost of more feed-forward operations. Finally we should discuss the size of the weak cross-Kerr nonlinearity required. Previously we have specified a constraint that $\alpha\theta^2 \sim 9$. Thus for realistic pumps with mean photon number per pulse on the order of 10^{12} (corresponding to $\alpha \sim 10^6$), a weak nonlinearity of the order of $\theta = 3 \times 10^{-2}$ should be sufficient to satisfy $\alpha\theta^2 \sim 9$. While this is still a technological challenge, it is likely to be achievable in the near future and really shows the potential power of weak (but not tiny) cross-Kerr nonlinearities. Strong nonlinearities are not a prerequisite to be able to perform quantum computation.

To summarize, we have shown in this Letter that weak cross-Kerr nonlinearities can be used to construct near deterministic CNOT gates with far fewer physical resources than other linear optical schemes. This has enormous implementations for the development of single photon quantum computing and information processing using either the convenient models or cluster state techniques. It can be immediately applied to optical cluster state computer allowing a significant reduction in the physical resources. At the core of the scheme are generalized QND detectors that allow us to distinguish subspaces of the basis states rather than all the basis states which occurs with the classic photon counters. The strength of the nonlinearities required for our gate are orders of magnitude weaker than those required to perform CNOT gates naturally between the single photons. Such nonlinearities are potentially available today using doped optical fibers, cavity QED, and EIT. We hope this work motivates the search for weak cross-Kerr nonlinearities which now have applications beyond, for instance, single photon number resolving detectors.

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